A challenging trig limit where L'Hôpital's Rule isn't so useful

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When L'Hôpital's Rule isn't the savior that it usually is, slick algebra can help.

Question.

Calculate:

$$\lim_{x \to 0} x^2 cot(8x) csc(7x)$$

Solution.

This limit contains a trap, and, "Substitute zero for x, and x^2 gives zero, so the limit is zero" springs it. The limit is actually not zero.

This logic would normally work, if it weren't for the *fractions* hiding out in this innocent-looking expression. The correct reasoning is that, when you substitute zero for x - in a rational expression - you get a numerator of zero, ***and a denominator of <u>not</u> zero*** - *that's* when it's appropriate to say that the limit is zero. For this limit, however, more work needs to be done. Algebra.

Using basic trigonometric identities,[1] let's rephrase this limit to see what's actually happening here:

$$\begin{aligned} L &= \lim_{x \to 0} x^2 cot(8x) csc(7x) \\ &= \lim_{x \to 0} x^2 \cdot \frac{cos(8x)}{sin(8x)} \cdot \frac{1}{sin(7x)} \end{aligned}$$

Now, it's clear that the denominator has to be zero at $x = 0$; more work is needed. In keeping with the title of this article, I will show the L'Hôpital's Rule method, but only after solving with a bit of a trick that gets the answer much quicker.

Please recall a basic trig limit result:

$$\lim_{x \to 0} \frac{sinx}{x} = 1$$

That is, for very very small values of x (the *angle*), the value of $sinx$ is equal to roughly that of the angle itself. We can use this sort of reasoning to also posit:

$$\lim_{x \to 0} \frac{x}{sinx} = 1 \tag{1}$$

as a corollary. This is also true. Another, more general corollary that follows is:

$$\lim_{x \to 0} \frac{ax}{sin(ax)} = 1 \tag{2}$$

[1] `https://en.wikipedia.org/wiki/List_of_trigonometric_identities` has a fantastic list of trig identities

for any constant a.

And now, for the trick. Let's rephrase our limit yet again:

$$L = \lim_{x \to 0} x^2 \cdot \frac{cos(8x)}{sin(8x)} \cdot \frac{1}{sin(7x)}$$

$$= \lim_{x \to 0} cos(8x) \cdot \frac{x}{sin(8x)} \cdot \frac{x}{sin(7x)} \qquad \text{(rephrase, cleverly)}$$

Notice that it sure would be nice if we could get those x's in the numerators to be $8x$ and $7x$, respectively, so we could use (2), above... Turns out, we can, if we do some slick algebra:

$$L = \lim_{x \to 0} cos(8x) \cdot \frac{x}{sin(8x)} \cdot \frac{x}{sin(7x)}$$

$$= \lim_{x \to 0} cos(8x) \cdot \frac{1}{8} \cdot \frac{8x}{sin(8x)} \cdot \frac{1}{7} \cdot \frac{7x}{sin(7x)} \qquad \text{(multiply terms by ... 1!)}$$

$$= \left(\lim_{x \to 0} cos(8x)\right)\left(\lim_{x \to 0} \frac{1}{8} \cdot \frac{8x}{sin(8x)}\right)\left(\lim_{x \to 0} \frac{1}{7} \cdot \frac{7x}{sin(7x)}\right) \qquad \text{(limit multiplication rule)}$$

$$= 1 \cdot \frac{1}{8} \cdot \frac{1}{7}$$

$$= \frac{1}{56}$$

Definitely not zero.

A look at Lôpital's Rule. Lôpital's Rule is usually a savior in situations like this. We are taking a limit, we substitute the value of x that we're approaching, we get an indeterminate form. Slam dunk. Let's actually have a look at what happens with L'Hôpotal's Rule in this case.

$$L = \lim_{x \to 0} x^2 \cdot \frac{cos(8x)}{sin(8x)} \cdot \frac{1}{sin(7x)}$$

$$= \lim_{x \to 0} \frac{x^2 cos(8x)}{sin(7x) sin(8x)} \qquad \text{(make one big happy fraction)}$$

We do get an indeterminate form when substituting, so we can apply L'Hôpital's Rule:

$$L = \lim_{x \to 0} \frac{x^2 cos(8x)}{sin(7x) sin(8x)}$$

$$= \lim_{x \to 0} \frac{2x cos(8x) - 8x^2 sin(8x)}{7 cos(7x) sin(8x) + 8 cos(8x) sin(7x)} \qquad \text{(derivative of the top over derivative of the bottom)}$$

Let's pause and see where we are. This limit got a bit more obfuscated than the limit we started with, due mostly to the products in the top and bottom of the fraction. Recall that this is a consequence of the product rule for derivatives:

$$\frac{d}{dx}\big[f(x) \cdot g(x)\big] = f'(x)g(x) + g'(x)f(x)$$

Specifically and practically, that from *one* product term comes *two* terms that we add together. This is borne out in that one application of L'Hôpital's Rule, above.

We see that we will still get an indeterminate form when we substitute 0 for x, so we can apply L'ôpital's Rule again. But. Now we have *four products, total* in our fraction: two on top, two on bottom, *each* of which will generate a sum of two terms. This will yield a whopping ***eight*** terms to get right *and* to manage as we move toward solution. Doesn't look fun.

Knowing what we know about what the answer actually is ($\frac{1}{56}$), we see that the only way that's going to happen is to knock out all of those x terms in the numerator, which means *at least* two more applications of L'Hôpital's Rule to completely knock out all the x's in the numerator. But, it gets worse. Since the x^2 is trapped in a product, it takes *twice as many derivatives* to get it down to a constant. Verify this for yourself by taking all those derivatives again, and see that the x^2 persists in that iteration of L'Hôpital's Rule. Moreover, since we are dealing with sines and cosines in the denominator, they are just going to produce more products of sine and cosine with ballooning constants in front of each pair. It is seriously messy with even one more application of L'Hôpital's Rule. I think you'll want to ditch this approach, too?

Done.

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