A tricky trigonometric integral with a surprising solution

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This is one that shows that things play out in interesting ways, at times.

Question.

Integrate:

$$I = \int \frac{dx}{sinx + sin2x}$$

Solution.

Ah, those two sine terms, so *un*like... We usually would like to do some sort of u-substitution to reconcile the x and $2x$ parts of the sine terms, but we lack the fundamental *other* requirement for u-substitutions: the derivative of the u we choose showing up in the integrand already, differing, at most, by a constant. Alas, there are no cosines ... yet. Time for Calculus Thinking.[1]

I honestly would have never thought to do this, but it works. Start by multiplying our integrand by $\frac{sinx}{sinx}$:

$$\int \frac{dx}{sinx + sin2x} = \int \frac{sinx\ dx}{sinx(sinx + sin2x)}$$

$$= \int \frac{sinx\ dx}{sinx(sinx + 2\ sinx\ cosx)} \qquad \text{(use trig identity for } sin2x\text{)}$$

$$= \int \frac{sinx\ dx}{sin^2x + 2\ sin^2x\ cosx} \qquad \text{(distribute)}$$

$$= \int \frac{sinx\ dx}{sin^2x(1 + 2\ cosx)} \qquad \text{(factor)}$$

$$= \int \frac{sinx\ dx}{(1 - cos^2x)(1 + 2\ cosx)} \qquad \text{(use Pythagorean identity)}$$

Now we have some cosines and *now* we can do a u-substitution:

[1] https://mymathteacheristerrible.com/blog/2019/5/15/calculus-thinking-a-definition-i-use-over-and-over-with-my-students

Let:

$$u = cosx$$
$$du = -sinx\ dx$$
$$dx = -\frac{du}{sinx}$$

Substituting, we have:

$$I = -\int \frac{du}{(1-u^2)(1+2u)} \qquad \text{(substitute)}$$

$$= \int \frac{du}{(u^2-1)(1+2u)} \qquad \text{(rearrange, factor out a minus sign)}$$

$$= \int \frac{du}{(u+1)(u-1)(1+2u)} \qquad \text{(factor denominator)}$$

For the astute observer, we have transformed this trig integral into one with a polynomial denominator suitable for *partial fraction expansion*. Super interesting. Let's proceed:

$$\frac{1}{(u+1)(u-1)(1+2u)} = \frac{A}{u+1} + \frac{B}{u-1} + \frac{C}{2u+1} \qquad \text{(linear factors, integer numerators)}$$

$$\begin{aligned} 1 &= A(u-1)(2u+1) + B(u+1)(2u+1) + C(u+1)(u-1) \\ &= A(2u^2-u-1) + B(2u^2+3u+1) + C(u^2-1) && \text{(expand)} \\ &= 2Au^2 - Au - A + 2Bu^2 + 3Bu + B + Cu^2 - C && \text{(distribute)} \\ &= (2A+2B+C)u^2 + (-A+3B)u + (-A+B-C) && \text{(collect like terms)} \end{aligned}$$

Collecting like terms and equating coefficients, this yields a system of 3 equations in 3 variables, A, B, and C, and it's actually not too bad:

$$-A + B - C = 1$$
$$2A + 2B + C = 0$$
$$-A + 3B + 0C = 0$$

We see that if we add up all 3 equations, we get $6B = 1$, so B is $\frac{1}{6}$.
Using the third equation, we get $-A + \frac{1}{2} = 0$, and that results in $A = \frac{1}{2}$.
Finally, using the first equation, we have $-\frac{1}{2} + \frac{1}{6} - C = 1$, and that gives $C = -\frac{4}{3}$.

Our integral then becomes:

$$I = \int \frac{1}{2(u+1)} + \frac{1}{6(u-1)} - \frac{4}{3(2u+1)}\ du$$

$$= \frac{1}{2}\int \frac{du}{(u+1)} + \frac{1}{6}\int \frac{du}{(u-1)} - \frac{4}{3}\int \frac{du}{(2u+1)} \qquad \text{(apply linearity and factor out constants)}$$

$$= \frac{1}{2}ln|u+1| + \frac{1}{6}ln|u-1| - \frac{2}{3}ln|2u+1| + C \qquad \text{(integrate)}$$

$$I = \frac{1}{2}ln|cosx+1| + \frac{1}{6}ln|cosx-1| - \frac{2}{3}ln|2cosx+1| + C \qquad \text{(rewrite in terms of } x\text{)}$$

Optional at this point is to use the rules of logarithms to consolidate this expression - would turn out to be a fraction with a product in the numerator. Also note how a pretty simplified integrand led to a fairly complicated antiderivative - interesting.

Done.

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