A tough trigonometric integral, solved multiple ways

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There are many ways to solve most problems. This is yet another one from Papa Flammy,[1] and with this one, I wondered if I could take another approach, using some *other* tricks. Turns out, I could. Here's how.

Question.

Integrate:

$$\int_0^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x} dx$$

Solution.

1 The master solves it, no comment needed

Papa Flammy did this integral using the Weierstrass substitution; it's quite elegant. Please watch, like, and subscribe: `https://www.youtube.com/watch?v=Uxd8MhECOWE`.

2 Another way: Odd/even properties and trig identities

Whenever I see an integral involving (strictly?) sines and cosines, I take a moment to think about how we might take advantage of the odd/even properties of sine and cosine and how they could be composited to form the function in question, namely, our integrand.

I covered this idea in a previous article here: **Crazy integration! Odd functions! Even functions! Calculus Thinking!**. Please have a look to brief yourself on the concepts.

In summary: Functions can be broken down into their odd and even components using 2 simple formulas. Over a symmetric interval, $x \epsilon [-a, a]$, we need only consider the *even* portion of any function, as integrating an odd function over $x \epsilon [-a, a]$ is zero. This is the basis of this approach.

The first step is to make sure we have a symmetric interval over which we wish to integrate, and, since our integral is over $x \epsilon [0, \frac{\pi}{2}]$, a simple slide to the left of $\frac{\pi}{4}$ is needed. We can accomplish this very easily:

[1] Flammable Maths channel: `https://www.youtube.com/channel/UCtAIs1VCQrymlAnw3mGonhw`

$$f(x) = \frac{sin^2\ x}{sin\ x + cos\ x} \qquad \text{(our integrand)}$$

$$f(x+\frac{\pi}{4}) = \frac{sin^2(x+\frac{\pi}{4})}{sin(x+\frac{\pi}{4}) + cos(x+\frac{\pi}{4})} \qquad \text{(side to the left by } \frac{\pi}{4}\text{)}$$

We can do a lot of algebra on this to simplify, taking advantage of trig identities to do so:

$$f(x+\frac{\pi}{4}) = \frac{sin^2(x+\frac{\pi}{4})}{sin(x+\frac{\pi}{4}) + cos(x+\frac{\pi}{4})}$$

$$= \frac{[sinx \cdot cos\frac{\pi}{4} + cosx \cdot sin\frac{\pi}{4}]^2}{sinx \cdot cos\frac{\pi}{4} + cosx \cdot sin\frac{\pi}{4} + cosx \cdot cos\frac{\pi}{4} - sinx \cdot sin\frac{\pi}{4}} \qquad \text{(expand, using angle sum identities)}$$

$$= \frac{(\frac{\sqrt{2}}{2})^2 \cdot (sinx + cosx)^2}{\frac{\sqrt{2}}{2} \cdot 2cosx} \qquad \text{(simplify - a tad tricky)}$$

$$= \frac{\sqrt{2}}{4} \cdot \frac{sin^2x + 2sinx \cdot cosx + cos^2x}{cosx} \qquad \text{(simplify a bit more, expand numerator)}$$

$$= \frac{\sqrt{2}}{4} \cdot \frac{1 + 2sinx \cdot cosx}{cosx} \qquad \text{(simplify, using Pythagorean identity)}$$

$$= \frac{\sqrt{2}}{4}\left(\frac{1}{cosx} + 2sinx\right) \qquad \text{(split up the fraction)}$$

$$f(x+\frac{\pi}{4}) = \frac{\sqrt{2}}{4}(secx + 2sinx) \qquad \text{(simplify, using reciprocal identity)}$$

Now we are ready to slide our entire integral to the left:

$$\int_0^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x}dx = \int_{0-\frac{\pi}{4}}^{\frac{\pi}{2}-\frac{\pi}{4}} \frac{\sqrt{2}}{4}(secx + 2sinx)\ dx \qquad \text{(substitute, transform limits of integration)}$$

$$= \frac{\sqrt{2}}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (secx + 2sinx)\ dx \qquad \text{(factor and simplify)}$$

$$= \frac{\sqrt{2}}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} secx\ dx + \frac{\sqrt{2}}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} sinx\ dx \qquad \text{(apply linearity of the integral, simplify)}$$

This is the kind of result we are after. As mentioned above, the integral of an odd function over a symmetrical interval about the origin is zero. Therefore, the second integral - the one involving $sinx$ - vanishes, leaving us with:

$$\int_0^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x}dx = \frac{\sqrt{2}}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} secx\ dx$$

Furthermore, note that $secx$ is an *even* function that we wish to evaluate over a symmetrical interval about the origin. We can apply a simplification here, as well, and solve:

$$\frac{\sqrt{2}}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} secx\ dx = 2\cdot\frac{\sqrt{2}}{4}\int_{0}^{\frac{\pi}{4}} secx\ dx \qquad \text{(use symmetry to reevaluate the limits)}$$

$$= \frac{\sqrt{2}}{2}\int_{0}^{\frac{\pi}{4}} secx\ dx \qquad \text{(simplify coefficient)}$$

$$= \frac{\sqrt{2}}{2} ln|secx + tanx|\Big]_{0}^{\frac{\pi}{4}} \qquad \text{(integrate)}$$

$$= \frac{\sqrt{2}}{2}\left[ln\left|sec\frac{\pi}{4} + tan\frac{\pi}{4}\right| - ln|sec0 + tan0|\right] \qquad \text{(evaluate antiderivative, using limits of integration)}$$

$$= \frac{\sqrt{2}}{2}(ln|\sqrt{2}+1| - ln|1+0|)$$

$$= \frac{\sqrt{2}}{2} ln(\sqrt{2}+1)$$

We have shown:

$$\int_{0}^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x}dx = \frac{\sqrt{2}}{2} ln(\sqrt{2}+1)$$

3 A third way: The King property and Weierstrass substitution

As with the last section, there are prerequisites for these techniques:

- Please see my article, **The King property of integration - discussion and 3 examples!**
- Please watch another master - blackpenredpen[2] - discuss the technique and results of the Weierstrass substitution, please watch, like, and subscribe: `https://www.youtube.com/watch?v=ggSrlWfbXC8`

We will use the results of both here.

Recall our integral, and let's call it I:

$$I = \int_{0}^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x}dx$$

Applying the King property, we have:

$$\int_{0}^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x}dx = \int_{0}^{\frac{\pi}{2}} \frac{sin^2(0+\frac{\pi}{2}-x)}{sin(0+\frac{\pi}{2}-x) + cos(0+\frac{\pi}{2}-x)}dx \qquad \text{(substitute)}$$

[2]blackpenredpen channel: `https://www.youtube.com/user/blackpenredpen`

$$= \int_0^{\frac{\pi}{2}} \frac{sin^2(\frac{\pi}{2} - x)}{sin(\frac{\pi}{2} - x) + cos(\frac{\pi}{2} - x)} dx \qquad \text{(simplify)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{cos^2\ x}{cosx + sinx} dx \qquad \text{(simplify, using cofunction identities)}$$

This only looks *slightly* different than what we started with, but note that this integral *also* represents I, giving us *two* definitions for I:

$$I = \int_0^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x} dx$$

and

$$I = \int_0^{\frac{\pi}{2}} \frac{cos^2\ x}{cosx + sinx} dx$$

Now, check out what happens when we add these two integrals together!

$$I + I = \int_0^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x} dx + \int_0^{\frac{\pi}{2}} \frac{cos^2\ x}{cosx + sinx} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x} + \frac{cos^2\ x}{cosx + sinx} dx \qquad \text{(simplify, apply linearity of the integral)}$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{sin^2\ x + cos^2\ x}{sin\ x + cos\ x} dx \qquad \text{(simplify, combine fractions)}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{sin\ x + cos\ x}\ dx \qquad \text{(simplify, using Pythagorean identity)}$$

Super cool, although, the experienced Calculus student might notice that it's really the *denominator* that will give us some trouble now. This is where the Weierstrass substitution can help. Per blackpenredpen's discussion, the first thing that has to happen is to let

$$t = tan\left(\frac{x}{2}\right)$$

making

$$x = 2tan^{-1}t$$

As far as our denominator is concerned, the other parts of this technique yield:

$$sinx = \frac{2t}{t^2 + 1}$$

and

$$cosx = \frac{1 - t^2}{t^2 + 1}$$

Transforming dx, we have

$$dx = \frac{2}{t^2 + 1} dt$$

One more thing! Under this transformation, limits of integration now become: $0 \to 0$ and $\frac{\pi}{2} \to 1$. With all of the components of our integral transformed, plugging in, we can now say:

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{sin\ x + cos\ x}\ dx \qquad \text{(Integral in } x\text{)}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{\frac{2t}{t^2+1} + \frac{1-t^2}{t^2+1}} \cdot \frac{2}{t^2+1}\ dt \qquad \text{(substitute)}$$

$$= \int_0^1 \frac{dt}{-t^2+2t+1} \qquad \text{(simplify - cleans up quite a bit!)}$$

$$= -\int_0^1 \frac{dt}{t^2-2t-1} \qquad \text{(factor denominator for completing the square)}$$

$$I = -\int_0^1 \frac{dt}{(t-1)^2-2} \qquad \text{(perform completing the square in denominator)}$$

From here, we can perform a u-substitution, letting $u = t - 1$. We then need to do partial fraction expansion on the integrand in u, leaving us, finally, with

$$I = \frac{\sqrt{2}}{2} ln(\sqrt{2}+1)$$

.

Again, have shown:

$$\int_0^{\frac{\pi}{2}} \frac{sin^2\ x}{sin\ x + cos\ x} dx = \frac{\sqrt{2}}{2} ln(\sqrt{2}+1)$$

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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Thank you so much.