Integrals of the form $\dfrac{1}{a+\sqrt{bx}}$

Phil Petrocelli, mymathteacheristerrible.com

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An application of u-substitution with a little twist.

Question.

Solve the following:

$$\int \frac{1}{1+\sqrt{2x}}\;dx$$

using $u = 1 + \sqrt{2x}$

Solution.

When I first saw this integral, I did not really know what to do with it. First thing that came to mind was the conjugate trick that we normally use for such things. I thought maybe I'd be able to make some sort of *arctan* thing out of it, but that quickly seemed to be a dead end. I took the suggestion that the book made:

$$\begin{aligned} u &= 1+\sqrt{2x} && (1)\\ &= 1+(2x)^{\frac{1}{2}} && (2) \end{aligned}$$

$$\begin{aligned} du &= (2x)^{-\frac{1}{2}}\;dx && \text{(remember to use the chain rule here)} && (3)\\ dx &= (2x)^{\frac{1}{2}}\;du && \text{(solve for } dx\text{, in preparation for substitution)} && (4) \end{aligned}$$

Well, that looked rather nasty to me at first blush, as it appeared to introduce more x's, and that's generally a signal that u-substitution is a fail for that particular integrand. I did push ahead and did the substitution the long way, and it still did not appear to be good. Here it is:

$$\int \frac{1}{1+\sqrt{2x}}\;dx = \int \frac{1}{u}\cdot(2x)^{\frac{1}{2}}\;du \qquad (5)$$

At this point, I was not convinced of this even working. I had to do *something* with that $(2x)^{\frac{1}{2}}$ term to continue! The key is to notice what's happening in (2) and (4), above:

$$\begin{aligned} u &= 1+(2x)^{\frac{1}{2}}\\ dx &= (2x)^{\frac{1}{2}}\;du \end{aligned}$$

If we solve (2) for $(2x)^{\frac{1}{2}}$, oddly enough, we get:

$$u-1 = (2x)^{\frac{1}{2}}$$

and substituting into (5), we now have:

$$\begin{aligned}
\int \frac{1}{1+\sqrt{2x}}\ dx &= \int \frac{1}{u} \cdot (2x)^{\frac{1}{2}}\ du \\
&= \int \frac{1}{u}(u-1)\ du \\
&= \int \frac{(u-1)}{u}\ du \\
&= \int\ du - \int \frac{du}{u} \\
&= u - ln|u| + C
\end{aligned}$$

Note how quickly and easily the integration went once we did that!

Back to the x-world, we have our final solution:

$$\int \frac{1}{1+\sqrt{2x}}\ dx = 1+\sqrt{2x} - ln|1+\sqrt{2x}| + C$$

Remember, we can always differentiate our answer to make sure we get the original integrand. Please verify.

Further discussion.

We can generalize the result of this particular integral by considering:

$$\int \frac{1}{a+\sqrt{bx}}\ dx, \text{ where } a \text{ and } b \text{ are constants}$$

While not really a fundamental integral, it is still useful to see what happens, in order to gather some insights about how to deal with integrals of this type. We begin with a more general u, analogous to what we did for the specific integral, above:

$$\begin{aligned}
u &= a + \sqrt{bx} \\
&= a + (bx)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
du &= \frac{1}{2}(bx)^{-\frac{1}{2}} \cdot b\ dx && \text{(chain rule)} \\
&= \frac{b}{2}(bx)^{-\frac{1}{2}}\ dx
\end{aligned}$$

And, remembering our trick,

$$du = \frac{b}{2}(bx)^{-\frac{1}{2}}\ dx$$

$$dx = \frac{2}{b}(bx)^{\frac{1}{2}}\ du \qquad \text{(solve for } dx\text{)}$$

$$u = a + (bx)^{\frac{1}{2}}$$

$$(bx)^{\frac{1}{2}} = u - a \qquad \text{(solve for } (bx)^{\frac{1}{2}})$$

And massaging our dx with the above, we get:

$$dx = \frac{2}{b}(bx)^{\frac{1}{2}}\ du$$

$$= \frac{2}{b}(u - a)\ du$$

So our integral becomes:

$$\frac{2}{b}\int \frac{u-a}{u}\ du$$

Solving, we have:

$$\frac{2}{b}\int \frac{u-a}{u}\ du = \frac{2}{b}\left[\int\ du - \int \frac{a}{u}\ du\right]$$

$$= \frac{2}{b}\left[u + a \cdot ln|u|\right] + C$$

And back to x, our answer for the general integral becomes:

$$\int \frac{1}{a+\sqrt{bx}}\ dx = \frac{2}{b}\left[a + \sqrt{bx} + a \cdot ln\left|a + \sqrt{bx}\right|\right] + C$$

Super interesting.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

My email address is: phil.petrocelli@gmail.com.

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