A really great related rates problem from the College Board

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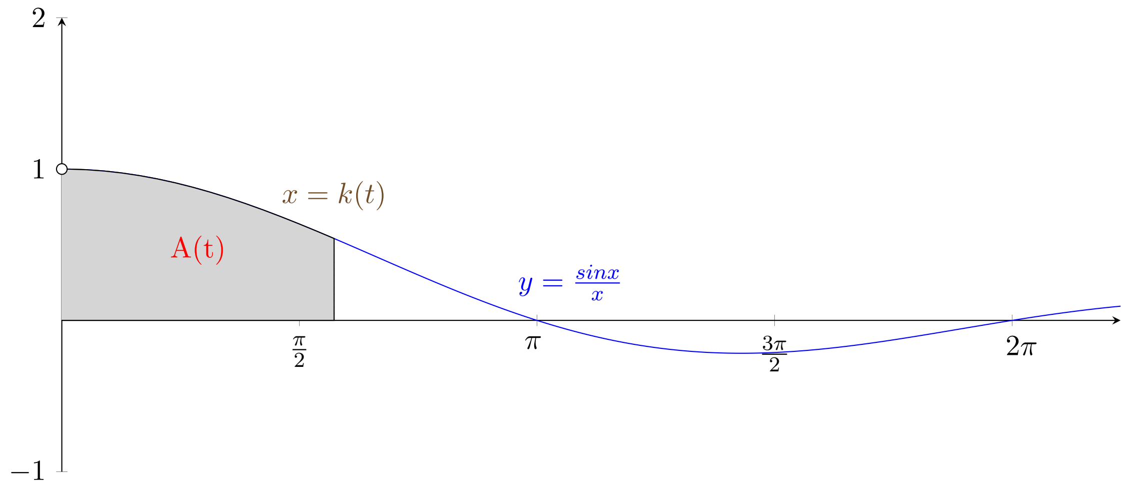
A student brought this to a tutoring session recently. It's a really great question from the AP Calculus Exam prep materials.

Question.



The figure shows region A, which is bounded by the x- and y-axes, and the graph of $f(x) = \dfrac{sinx}{x}$, for $x > 0$, and the vertical line $x = k$. If k increases at a rate of $\dfrac{\pi}{4}$ units per second, how fast is the area of region A changing when $k = \dfrac{\pi}{6}$?

Solution.

This is really a *related rates problem with an integral* - *super* interesting! Let's have a look at what's being said and how we can begin to solve this problem.

The first thing to note is that we have a function, $f(x) = \frac{sinx}{x}$ that gives us the bounds of an area we need to compute. We can come up with an integral - that is a function, let's call it $A(t)$, that denotes the area in question, no problem. What's interesting about this integral is that the upper bound is *moving*.

The phrase "... k increases at a rate of..." also signals that k is some function of t and has an associated rate, and whenever we say "rate" in Calculus, we mean "instantaneous rate of change" or, in terms of calculation, we mean ***derivative***. Let's call the function $k(t)$ and call its first derivative, $\frac{d}{dt}k$, or $\frac{dk}{dt}$. With this in mind, we can say our area, as a function of t, is:

$$A(t) = \int_0^{k(t)} \frac{sinx}{x}\, dx$$

On closer inspection of the integrand and of the graph of $f(x)$, we have a domain issue, namely x cannot equal zero, so, technically, $A(t)$ is an improper integral, and should be stated as:

$$A(t) = \int_{0^+}^{k(t)} \frac{sinx}{x}\, dx \qquad \text{(note the plus sign in the lower bound)} \tag{1}$$

We can write the rest of the given information as:

$$k(t) = \frac{\pi}{6} \tag{2}$$

$$\frac{dk}{dt} = \frac{\pi}{4} \text{ units/sec} \tag{3}$$

This question is really asking about the rate at which $A(t)$ is changing for a given value of $k(t)$ at time t, so we must differentiate (1):

$$A(t) = \int_{0^+}^{k(t)} \frac{sinx}{x}\, dx$$

$$\frac{d}{dt}A(t) = \frac{d}{dt}\int_{0^+}^{k(t)} \frac{sinx}{x}\, dx \tag{4}$$

Now, the phrasing of (4) is in classic Fundamental-Theorem-of-Calculus form. Please recall that part of the Fundamental Theorem of Calculus establishes differentiation and integration as *inverse* operations; this will help here. Also take a moment to see that $A(t)$ being an improper integral and being an integral we have to calculate - at all - is not part of what's being asked here. We are merely interested in the rate of change of $A(t)$, namely, $A'(t)$, when $k(t) = \frac{\pi}{6}$. Differentiating, we have:

$$\begin{aligned}
\frac{d}{dt}A(t) &= \frac{d}{dt}\int_{0^+}^{k(t)} \frac{sinx}{x}\, dx \\
A'(t) &= \frac{sin[k(t)]}{k(t)} \cdot \frac{dk}{dt} && \text{(note the chain rule, since } k \text{ is a function of } t\text{!!!)} \\
&= \frac{sin\frac{\pi}{6}}{\frac{\pi}{6}} \cdot \frac{\pi}{4} \text{ units}^2\text{/sec} && \text{(substitute)} \\
&= \frac{1}{2} \cdot \frac{6}{\pi} \cdot \frac{\pi}{4} \text{ units}^2\text{/sec} && \text{(simplify)} \\
&= \frac{3}{4} \text{ units}^2\text{/sec} && \text{(simplify)}
\end{aligned}$$

This is a really great question that tests a few skills/calculations simultaneously. Please visit `https://ap.collegeboard.org/` for more information on the AP exam program, including the exams for Calculus and Statistics.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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Thank you so much.