A curve sketching question that hits all the finer points of the technique

Phil Petrocelli, mymathteacheristerrible.com

November 16, 2019

How do I know that I understand the finer points of curve sketching using Calculus I techniques? Common question. Here is a discussion of a problem that has it all; if you can handle this one, you've got it.

Question.

Find all intercepts, relative extrema, inflection points, and asymptotes of:

$$f(x) = \frac{x^3}{x^2 - 9}$$

Solution.

This sounds an awful lot like an Algebra II problem. It sort of is, but by the time you get a question like this in Calculus, you have learned a few general techniques that make graphing *any* function easier. We can, however, use some Algebra II techniques to verify what's happening with this function.

For ease of reading when we take derivatives, let's say:

$$y = \frac{x^3}{x^2 - 9}$$

to begin working on this one. Let's also note that this is not the only form for this function, specifically, note:

$$y = \frac{x^3}{(x+3)(x-3)} \qquad \text{(factored form in denominator)} \tag{1}$$

$$y = x + \frac{9x}{x^2 - 9} \qquad \text{(result of long division)} \tag{2}$$

$$y = x + \frac{9x}{(x+3)(x-3)} \qquad \text{(long division \textbf{and} factored form of denominator)} \tag{3}$$

If you are a regular reader of my articles, you know I discuss not only *calculations*, but ***managing*** *calculations*, as well. Noting the above 3 expressions of the same function is a key management point for this problem. As you will see, we will fluidly switch between these four forms of our function, depending on context and ease of calculations.

Vertical asymptotes. This is not a Calculus topic. Recall that vertical asymptotes are where a rational function becomes undefined, specifically, where the denominator equals zero. Here is the first management point. If we use (1), we see that we can easily find the roots of the denominator:

$$(x-3)(x+3) = 0$$

Simply, $x = \pm 3$.

Horizontal and other asymptotes. This is *sort of* a Calculus topic. Recall that when we talk about horizontal and other asymptotes, in Calculus terms, we are talking about limits at infinity for the given function:

$$\lim_{x \to -\infty} f(x)$$

and

$$\lim_{x \to \infty} f(x)$$

Fortunately for us, this problem is a ratio of polynomials. Since this is the case, we have two options: we can use limits at infinity or we can use the *leading coefficient test.*[1] Since the degree of the numerator is greater than the degree of the denominator, we cannot use the leading coefficient test on the function as originally stated (we wind up with $\frac{1}{0}$), but if we use (2), we see that we can do the leading coefficient test on the fractional part and get $\frac{0}{1} = 0$. In addition, you may recall from Algebra II that the whole number part of (2) (i.e., not the fraction/remainder) is the expression for the asymptote. For this function, the asymptote is a slant asymptote: $y = x$, and for any line, $y = mx + b$, if $m > 0$, the line goes to $-\infty$ for negative x, and to $+\infty$ for positive x.

If we look at the graph of $f(x)$ (`https://www.desmos.com/calculator/e6gzsh3g8v`), we see that $y = x$ is indeed the (slant) asymptote, that is, it is $\lim_{x \to \pm\infty} f(x)$. Take a moment to start with $f(x)$ and use limits at infinity to see that you get the same results.

Note that there are no horizontal asymptotes for this function, just the slant asymptote, as discussed.

Intercepts. This is not a Calculus topic. Recall the definitions: y-intercepts are the values of the function when $x = 0$, and x-intercepts are the values of x when $y = 0$.

To find the y-intercept:

$$f(0) = \frac{0^3}{(0-3)(0+3)} \qquad \text{(substitute 0 for } x\text{)}$$

$$= 0 \qquad \text{(a fraction is zero when the numerator is zero and the denominator is not zero)}$$

So, the y-intercept is zero.

To find the x-intercepts:

$$f(x) = 0 \qquad \text{(substitute zero for } y\text{)}$$

$$\frac{x^3}{(x+3)(x-3)} = 0$$

$$x^3 = 0, \text{ if } x \neq \pm 3 \qquad \text{(simplify, with caveat)}$$

$$x = 0 \qquad \text{(solve for } x\text{)}$$

So, the x-intercept is zero.

And, with that, the non-Calculus parts of this problem are complete. Moving on...

Relative extrema. Recall that finding relative extrema begins with finding the critical points of the given function, and that critical points are where the first derivative of the function is zero or undefined. Let's take a

[1] I really wish Calculus teachers would point this out more often. The limits at infinity technique works, but often there is a shortcut, as this case demonstrates.

derivative:

$$y = \frac{x^3}{x^2 - 9}$$

$$y' = \frac{3x^2(x^2-9) - 2x \cdot x^3}{(x^2-9)^2} \qquad \text{(quotient rule)}$$

$$= \frac{3x^4 - 27x^2 - 2x^4}{(x^2-9)^2}$$

$$= \frac{x^4 - 27x^2}{(x^2-9)^2} \tag{4}$$

$$y' = \frac{x^2(x^2-27)}{(x^2-9)^2} \qquad \text{(simplify)} \tag{5}$$

Now, set the derivative equal to zero:

$$\frac{x^2(x^2-27)}{(x^2-9)^2} = 0$$

$$x^2(x^2-27) = 0 \qquad \text{(a fraction is zero when numerator is zero and denom is not zero)}$$

$$x = 0,\ \pm 3\sqrt{3}$$

Critical points are also where the derivative does not exist, and for this function, a zero in the denominator yields critical numbers of $x = \pm 3$, making

$$x = 0,\ \pm 3\sqrt{3},\ \pm 3$$

our set of critical points.

We could compute $f(x)$ for all of our critical points, but it might make more sense to use the second derivative test to evaluate and name extrema, so let's do that. Taking a second derivative, we have:

$$\frac{d^2}{dx^2}y = \frac{d}{dx}y'$$

$$y'' = \frac{d}{dx}\left[\frac{x^4 - 27x^2}{(x^2-9)^2}\right]$$

Another management opportunity lies here. Note that I choose to use (4), above, rather than the simplified version in (5). This is because, looking ahead to having to use the quotient rule to calculate y'', I'd love to have a simple polynomial in the numerator for ease of calculation. Note, also, that I did *not* choose to expand the denominator, as it does appear to be a clean-ish expression to differentiate using a simple application of the chain rule. Let's proceed:

$$y'' = \frac{d}{dx}\left[\frac{x^4 - 27x^2}{(x^2-9)^2}\right]$$

$$= \frac{(4x^3 - 54x)(x^2-9)^2 - 2 \cdot 2x(x^2-9)(x^4 - 27x^2)}{(x^2-9)^4} \qquad \text{(quotient rule, with chain rule for the denom)}$$

$$= \frac{(4x^3 - 54x)(x^2-9)^2 - 4x(x^2-9)(x^4-27x^2)}{(x^2-9)^4} \qquad \text{(simplify a little)}$$

$$= \frac{2x(x^2-9)\big[(2x^2-27)(x^2-9) - 2(x^4-27x^2)\big]}{(x^2-9)^4} \qquad \text{(pro-gamer-move}^2\text{ factoring)}$$

$$= \frac{2x(x^2-9)\big[2x^4 - 18x^2 - 27x^2 + 243 - 2x^4 + 54x^2\big]}{(x^2-9)^4} \qquad \text{(expand the numerator to see if it can clean up a bit)}$$

$$= \frac{2x(x^2-9)\big(9x^2+243\big)}{(x^2-9)^4} \qquad \text{(simplify numerator)}$$

$$= \frac{18x(x^2-9)\big(x^2+27\big)}{(x^2-9)^4} \qquad \text{(factor the numerator)}$$

$$y'' = \frac{18x\big(x^2+27\big)}{(x^2-9)^3} \qquad \text{(cancel terms in numerator \& denom)}$$

Setting y'' equal to zero and solving, we have:

$$y'' = 0$$

$$\frac{18x\big(x^2+27\big)}{(x^2-9)^3} = 0$$

This yields $x = 0, \pm 3$ as points to examine ($x = \pm 3$ is where y'' does not exist).

Calculations are *done*, so now we need to make sense of what's happening on which intervals:

Interval	y'	y''	Meaning
$(-\infty, -3\sqrt{3})$	positive	negative	increasing, concave down
$(-3\sqrt{3}, -3)$	negative	negative	decreasing, concave down
$(-3, 0)$	negative	positive	decreasing, concave up
$(0, 3)$	negative	negative	decreasing, concave down
$(3, 3\sqrt{3})$	negative	positive	decreasing, concave up
$(3\sqrt{3}, \infty)$	positive	positive	increasing, concave up

[2] With thanks to `https://www.youtube.com/user/Vsauce`

Using this table we can find our relative extrema:

Critical point	From	To	Meaning
$x = -3\sqrt{3}$	increasing, concave down	decreasing, concave down	relative max
$x = -3$	decreasing, concave down to $-\infty$	decreasing, concave up from $+\infty$	vertical asymptote
$x = 0$	decreasing, concave up	decreasing, concave down	inflection point
$x = 3$	decreasing, concave down to $-\infty$	decreasing, concave up from $+\infty$	vertical asymptote
$x = 3\sqrt{3}$	decreasing, concave up	increasing, concave up	relative min

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

My email address is: phil.petrocelli@gmail.com.

Please visit `https://mymathteacheristerrible.com` for other study guides. Please tell others about it.

Please donate

I write these study guides with interest in good outcomes for math students and to be a part of the solution. If you would consider donating a few dollars to me so that these can remain free to everyone who wants them, please visit my PayPal and pay what you feel this is worth to you. Every little bit helps.

My PayPal URL is: `https://paypal.me/philpetrocelli`.

Thank you so much.