Two ways to solve this (sort of) power-reduction Trig integral

Phil Petrocelli, mymathteacheristerrible.com

February 24, 2020

Often, there are many ways to solve the same integration problem. This is an interesting one I saw on Reddit (r/Calculus) recently.

Question.

Integrate:

$$\int_0^{\frac{\pi}{4}} tan^7 x\ sec^2 x\ dx$$

Solution.

To set up the rest of this discussion, let's name our integral I_1 and I_2, where $I_1 = I_2$:

$$I_1 = \int_0^{\frac{\pi}{4}} tan^7 x\ sec^2 x\ dx \qquad \text{(original integral)}$$

$$I_2 = \int_0^{\frac{\pi}{4}} tan^6 x\ secx\ secx\ tanx\ dx \qquad \text{(rewritten with a } secx \text{ and } tanx \text{ "peeled off")}$$

Now, both of these integrals are signaling u-substitutions, as both have a function and its derivative expressed in the integrand. Super interesting. Let's examine both ways to solve.

Solving I_1

Turns out, this one is the most straightforward one to solve, so it will go quicker. Starting with a u-substitution, let:

$$u = tanx$$
$$du = sec^2 x\ dx$$
$$dx = \frac{du}{sec^2 x}$$

$$a = tan(0) = 0$$
$$b = tan\left(\frac{\pi}{4}\right) = 1$$

Taking I_1 to the u-world, we now have:

$$I_1 = \int_0^1 u^7 \, du$$

$$= \frac{1}{8} u^8 \Big]_{u=0}^{1}$$

$$= \frac{1}{8} \cdot 1^8 - 0$$

$$= \frac{1}{8}$$

Not too bad. Done.

Solving I_2

This is an alternate solution, starting with an alternate choice for u:

$$u = secx$$
$$du = secx \; tanx \; dx$$
$$dx = \frac{du}{secx \; tanx}$$

$$a_u = sec(0) = 1$$
$$b_u = sec\left(\frac{\pi}{4}\right) = \sqrt{2}$$

In addition, we have to think about the $tan^6 x$ term. Here, we use the Pythagorean identity:

$$tan^2 x = sec^2 x - 1$$

giving us:

$$tan^6 x = (tan^2 x)^3 = (sec^2 x - 1)^3$$

Taking I_2 to the u-world, we now have:

$$I_2 = \int_1^{\sqrt{2}} (u^2 - 1)^3 \cdot u \, du$$

OK, let's pause and note that we have two options for proceeding. First is to expand the cubic term, $(u^2 - 1)^3$ and multiply by u to get a polynomial integral in u - super easy to solve, although the expansion is a bit painful. Second, note that we do have a term and its derivative expressed in the integrand of the u integral, namely, $(u^2 - 1)$ and its derivative, $2u$, where $2u$ differs from u only by a constant. I like the latter, myself, since it is a way to do the integral without introducing possible errors by expanding incorrectly, as well as it being a great example of Calculus Thinking.[1] Since we have already used u to substitute and rewrite I_2 once, let's use v to rewrite our u integral:

$$v = u^2 - 1$$

[1] https://mymathteacheristerrible.com/blog/2019/5/15/calculus-thinking-a-definition-i-use-over-and-over-with-my-students

$$dv = 2u\ du$$
$$du = \frac{dv}{2u}$$

$$a_v = 1^2 - 1 = 0$$
$$b_v = (\sqrt{2})^2 - 1 = 1$$

Taking I_2 to the v-world, we now have:

$$I_2 = \frac{1}{2}\int_0^1 v^3\ dv$$

$$= \frac{1}{2}\cdot\frac{1}{4}v^4\Big]_{v=0}^{1}$$

$$= \frac{1}{8}(1^4 - 0^4)$$

$$= \frac{1}{8}$$

Same answer, two ways! Yes!

It bears mentioning that neither solution is more correct than the other. One of the beautiful things I find in working math problems of all types is seeing new ways to solve old problems, and to track which paths to solution I see first. This is usually some sort of measure of my experience, and therefore, my intuition as a mathematician and problem solver. I check on both of those things quite often to see where I am, to see how I might better meet students in my work as a math tutor.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

My email address is: phil.petrocelli@gmail.com.

Please visit `https://mymathteacheristerrible.com` for other study guides. Please tell others about it.

Please donate

I write these study guides with interest in good outcomes for math students and to be a part of the solution. If you would consider donating a few dollars to me so that these can remain free to everyone who wants them, please visit my PayPal and pay what you feel this is worth to you. Every little bit helps.

My PayPal URL is: `https://paypal.me/philpetrocelli`.

Thank you so much.