L'Hôpital's Rule season continues!

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Another one, via Reddit (r/Calculus).

Question.

Compute:

$$\lim_{x\to\infty}(e^{3x}+5x)^{\frac{2}{x}}$$

Solution.

From the title of this article, you could probably guess that L'Hôpital's Rule will be needed. Let's start by the usual "substitution":

$$L = \lim_{x\to\infty}(e^{3x}+5x)^{\frac{2}{x}}$$

$$= (e^{3\cdot\infty}+5\cdot\infty)^{\frac{2}{\infty}}$$

$$= (\infty^2)^{\frac{1}{\infty}}$$

$$= (\infty)^{\frac{1}{\infty}}$$

Definitely an indeterminate form. So, let's try to get our limit into ratio form, in order to use L'Hôpital's Rule. Let's start with logarithmic differentiation:

$$L = \lim_{x\to\infty}(e^{3x}+5x)^{\frac{2}{x}}$$

$$ln\ L = ln\left[\lim_{x\to\infty}(e^{3x}+5x)^{\frac{2}{x}}\right] \qquad \text{(take the } ln \text{ of both sides)}$$

$$= \lim_{x\to\infty} ln\left[(e^{3x}+5x)^{\frac{2}{x}}\right] \qquad (ln \text{ is continuous on } x\to\infty\text{, can move the } ln \text{ inside the limit})$$

$$= \lim_{x\to\infty} \frac{2}{x}\ ln(e^{3x}+5x) \qquad \text{(power property of logs)}$$

$$ln\ L = 2 \cdot \lim_{x\to\infty} \frac{ln(e^{3x}+5x)}{x} \qquad \text{(clean things up to get into fraction form)}$$

Please verify for yourself that our limit, in this form, properly yields an indeterminate form. Now for the L'Hôpital's Rule step:

$$\begin{aligned} ln\ L &= 2 \cdot \lim_{x\to\infty} \frac{ln(e^{3x}+5x)}{x} \\ &= 2 \cdot \lim_{x\to\infty} \frac{\frac{1}{(e^{3x}+5x)} \cdot (3e^{3x}+5)}{1} && \text{(take the derivative of top and bottom)} \\ &= 2 \cdot \lim_{x\to\infty} \left(\frac{3e^{3x}+5}{e^{3x}+5x}\right) && \text{(simplify)} \end{aligned}$$

Please verify for yourself that we are not out of the woods, yet! This limit still results in an indeterminate form! Luckily, we can apply L'Hôpital's Rule again:

$$\begin{aligned} ln\ L &= 2 \cdot \lim_{x\to\infty} \left(\frac{3e^{3x}+5}{e^{3x}+5x}\right) \\ &= 2 \cdot \lim_{x\to\infty} \left(\frac{9e^{3x}}{3e^{3x}+5}\right) && \text{(take the derivative of top and bottom)} \\ &= 2 \cdot \frac{9}{3} && \text{(calculate limit by leading coeff test or other means)} \\ ln\ L &= 6 \\ L &= e^6 \end{aligned}$$

Having a look at the graph, `https://www.desmos.com/calculator/gstkrw7eip`, it shows that there is a horizontal asymptote at approximately 403. This is roughly e^6, as calculated.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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Thank you so much.