A challenging L'Hôpital's Rule limit

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Very interesting application of L'Hôpital's Rule, where we have a tough derivative that results in easing the situation.

Question.

Calculate:

$$\lim_{x\to\infty} x^{e-x}$$

Solution.

On the heels of having discussed this difficult limit: `https://mymathteacheristerrible.com/blog/2020/1/24/a-tough-limit-problem-two-ways`, I read about the above limit on Reddit. It's a tough one.

Although, as I discuss in `https://mymathteacheristerrible.com/blog/2019/2/24/free-calculus-study-guide-difficult-limits`, infinity is not a number, we can "substitute" and see what kind of expression we get:

$$\begin{aligned}\lim_{x\to\infty} x^{e-x} &= \infty^{e-\infty} \\ &= \infty^{-\infty}\end{aligned}$$

Or, if we have forgotten some of our indeterminate forms, we can rewrite first, using the rules of exponents, and *then* substitute:

$$\begin{aligned}\lim_{x\to\infty} x^{e-x} &= \lim_{x\to\infty} \frac{x^e}{x^x} && \text{(rewriting as a fraction is useful here)} \qquad (1) \\ &= \frac{\infty}{\infty^\infty} \\ &= \frac{\infty}{\infty}\end{aligned}$$

This is definitely a situation for L'Hôpital's Rule. As a reminder, when a limit, L, results in an indeterminate form, we need to rewrite the limit as the limit of a ratio, $\frac{f(x)}{g(x)}$, and then calculate: $\lim_{x\to a} \frac{f'(x)}{g'(x)}$. This is the value of our original limit, L.

Let's have a look at (1). Naming our $f(x)$ and $g(x)$ is a great first step:

$$f(x) = x^e$$
$$g(x) = x^x$$

L'Hôpital's Rule says we need some derivatives:

$$\begin{aligned} f(x) &= x^e \\ f'(x) &= e \cdot x^{e-1} \end{aligned} \tag{2}$$

The derivative of $g(x) = x^x$ is a little more difficult, but do-able:[1]

$$\begin{aligned}
g(x) &= x^x \\
y &= x^x \\
ln(y) &= ln(x^x) && \text{(set up for logarithmic differentiation)} \\
ln(y) &= xlnx && \text{(exponent rule for logs)} \\
\\
\frac{d}{dx}\Big[ln(y) &= xlnx\Big] && \text{(differentiate)} \\
\frac{1}{y}\frac{dy}{dx} &= lnx + x \cdot \frac{1}{x} && \text{(chain rule)} \\
\\
\frac{dy}{dx} &= y(lnx+1) \\
\\
&= x^x(lnx+1)
\end{aligned} \tag{3}$$

So, we can say:

$$g'(x) = x^x(lnx+1)$$

This makes our limit, L:

$$\begin{aligned}
L = &\lim_{x\to\infty} x^{e-x} \\
\\
&= \lim_{x\to\infty} \frac{x^e}{x^x} && \text{(fraction form of } L\text{)} \\
\\
&= \lim_{x\to\infty} \frac{e \cdot x^{e-1}}{x^x(lnx+1)} && \text{(apply L'Hôpital's Rule, using (2) and (3))} \\
\\
&= \lim_{x\to\infty} \frac{x^e}{x^x} \cdot \frac{e}{x(lnx+1)} && \text{(factor, cleverly)} \\
\\
&= \left(\lim_{x\to\infty} \frac{x^e}{x^x}\right) \cdot \left(\lim_{x\to\infty} \frac{e}{x(lnx+1)}\right) && \text{(limit product rule - note that } L \text{ appears!)} \\
\\
&= L \cdot \lim_{x\to\infty} \frac{e}{x(lnx+1)} && \text{(rewrite)}
\end{aligned}$$

[1] blackpenredpen has a great discussion at `https://www.youtube.com/watch?v=l-iLg07zavc`. Please check it out.

$$= L \cdot \left(\lim_{x \to \infty} \frac{e}{x} \right) \cdot \left(\lim_{x \to \infty} \frac{1}{lnx + 1} \right) \qquad \text{(apply limit product rule again)}$$

$$= L \cdot 0 \cdot 0 \qquad \text{(calculate limits)}$$

$$= 0$$

Please verify with Desmos[2] or any graphing calculator that 0 is the correct answer!

Done.

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[2] `https://www.desmos.com/calculator`

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