An implicit differentiation problem

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Implicit differentiation is very useful in a number of tricky situations. Here, we'll use implicit differentiation twice to solve a tough problem, noting some important choices we make along the way.

Question.

Find $\frac{d^2y}{dx^2}$, for:

$$x^4 + y^4 = 16$$

Solution.

Recall that $\frac{d^2y}{dx^2}$ is the Leibniz notation for y'' and $f''(x)$. These all mean the same thing. This notation also implies that y is a function of x.

We can solve for y here, but it might get a little messy when it comes time to take a derivative:

$$\begin{aligned} y &= \pm\sqrt[4]{16 - x^4} \\ &= \pm(16 - x^4)^{\frac{1}{4}} \end{aligned}$$

It's do-able, but we see, straight away, that we have to manage both the positive and negative parts of this expression separately[1], which means two different derivatives, and it sort-of escalates from there (especially when we have to differentiate *again* later). However, *implicit* differentiation is a lot cleaner:

$$\frac{d}{dx}\left[x^4 + y^4 = 16\right]$$

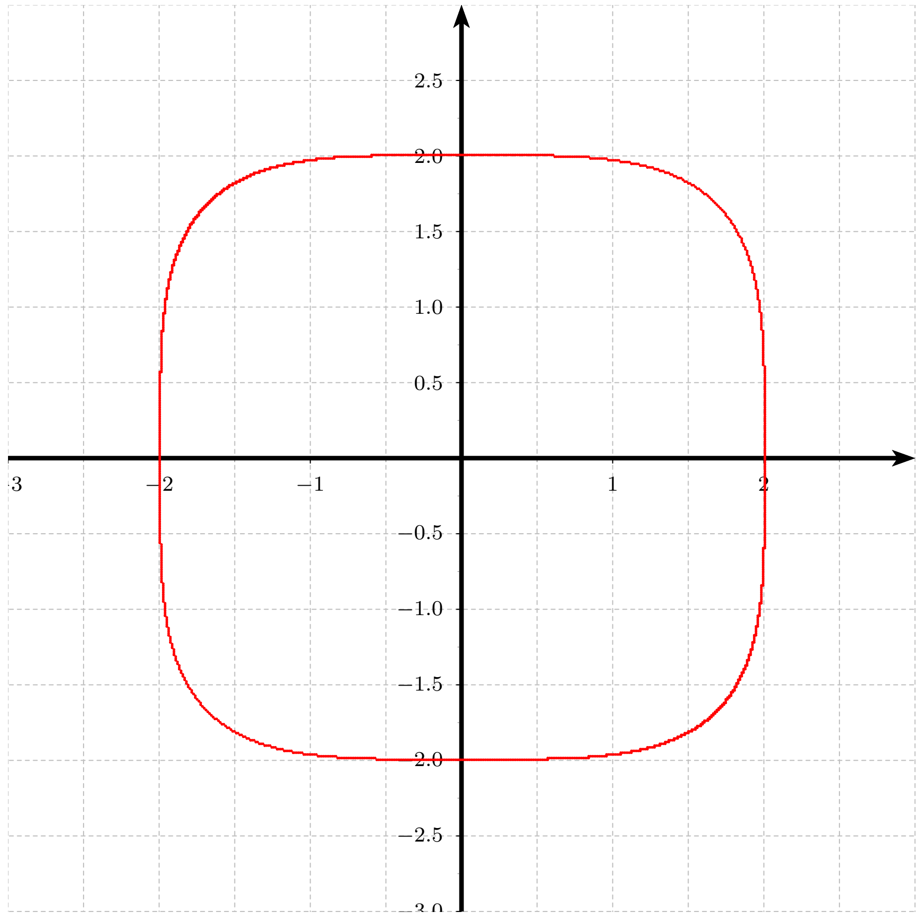
$$4x^3 + 4y^3\frac{dy}{dx} = 0 \qquad \text{(differentiate implicitly with respect to } x\text{)}$$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3} \qquad \text{(solve for } \frac{dy}{dx}\text{)}$$

$$= -\frac{x^3}{y^3} \qquad \text{(simplify)}$$

As a reminder, this says, that, for every point (x, y) *that is on the graph of* $x^4 + y^4 = 16$, the slope of the tangent at that point is $-\frac{x^3}{y^3}$. Have a look at the graph of our function to verify.

[1]Remember that relationships like this, stated with even powers of x and/or y, really consist of a *top half* function and a *bottom half* function graphed on the same set of axes, *each* of which passes the vertical line test. This is why the $\pm$ appears when we solve for x or y, and we have to carry the $\pm$ through the rest of our calculations.



NOTE: If you have covered conic sections in a previous course, you may recognize this approach, where we do not solve for y as a function of x.[2] Furthermore, this is where a graphing tool like Desmos (`https://www.desmos.com/calculator`) really comes in handy, as we can type in something like $x^4 + y^4 = 16$ and it will properly graph this figure.

Now, if we were to write $\frac{dy}{dx}$ as a function of x, we would have to substitute $(16 - x^4)^{\frac{1}{4}}$ for y, giving us:

$$\frac{dy}{dx} = -\frac{x^3}{\pm(16 - x^4)^{\frac{3}{4}}} \tag{1}$$

$$= \pm\frac{x^3}{(16 - x^4)^{\frac{3}{4}}} \tag{2}$$

We are only halfway done, however; we have to find $\frac{d^2y}{dx^2}$ to finish this problem. Note that the above form is a little complicated to differentiate, as it involves the quotient rule as well as the two halves of the figure, above the x-axis and below the x-axis. Ugh.

But here is where choices I mentioned come in. We can actually take the implicit expression for $\frac{dy}{dx}$ and differentiate *that* with respect to x to obtain $\frac{d^2y}{dx^2}$:

$$\frac{dy}{dx} = -\frac{x^3}{y^3}$$

[2]I discuss two conic sections in this article:
`https://mymathteacheristerrible.com/blog/2019/5/21/the-ellipse-hyperbola-connection-graphing-made-easy`

$$\frac{d}{dx}\left[\frac{dy}{dx} = -\frac{x^3}{y^3}\right]$$ (differentiate implicitly)

$$\frac{d^2y}{dx^2} = -\left(\frac{3x^2y^3 - 3x^3y^2\frac{dy}{dx}}{y^6}\right)$$ (Quotient rule:

Remember to use the chain rule when differentiating the denominator!)

$$= \frac{3x^3y^2\frac{dy}{dx} - 3x^2y^3}{y^6}$$ (distribute the minus sign)

$$= \frac{3x^2y^2(x\frac{dy}{dx} - y)}{y^6}$$ (factor the numerator)

$$= \frac{3x^2(x\frac{dy}{dx} - y)}{y^4}$$ (simplify)

$$= \frac{3x^2(x[-\frac{x^3}{y^3}] - y)}{y^4}$$ (substitute for $\frac{dy}{dx}$, first using the implicit expression)

$$= \frac{3x^2(-\frac{x^4}{y^3} - y)}{y^4}$$ (simplify)

$$= \frac{-3x^2\frac{1}{y^3}(x^4 + y^4)}{y^4}$$ (really cool factoring trick! Please work this one out for yourself!)

$$= \frac{-3x^2(x^4 + y^4)}{y^7}$$ (simplify)

$$= \frac{-3x^2(16)}{y^7}$$ (we know $x^4 + y^4$ from the statement of the problem)

$$\frac{d^2y}{dx^2} = -\frac{48x^2}{y^7}$$

This is a classic problem where implicit differentiation can make calculations a bit easier to manage. Of equal import is keeping in mind that, when we wind up with an expression for $\frac{d^n}{dx^n}$, it describes the *value* of the derivative at a point *on the given curve*. Should we want an expression that is a function of the independent variable, we can see that there are better times during solution than others to calculate this, as shown in (1), (2), above. This is a discussion of balancing our approach to calculations with an eye toward context and what I call ***calculus thinking***[3].

Done.

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[3] https://mymathteacheristerrible.com/blog/2019/5/15/calculus-thinking-a-definition-i-use-over-and-over-with-my-students

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