A great use of implicit differentiation: easy slopes of tangents

Phil Petrocelli, mymathteacheristerrible.com

October 26, 2019

We know that implicit differentiation helps in many situations where we are not able to easily separate the dependent variable from the independent variable (single-variable Calculus). This example problem reminds us of why this is so useful.

Question.

Find the equation of the tangent line to:

$$3(x^2 + y^2)^2 = 100(x^2 - y^2)$$

At the point $(4, 2)$

Solution.

There are a variety of ways of taking the derivative of the given relationship. If we have a look at `https://www.desmos.com/calculator/opwrxzpcif`, we can see that the graph is that of a polar-looking curve. From the graph, we also see that this curve is not a function, as-is, and we would need some massaging of the *implicitly* ***defined*** function, y, in order to find the equation of the tangent line at the given point. Since we have a given point on the graph, all we need is the *value* of the slope of the tangent at that point in order to come up with our equation using the point-slope formula for lines. As a reminder, the Calculus version, the equation of the tangent line to a curve at a point (x_1, y_1), is given by:

$$y - y_1 = y'|_{x=x_1}(x - x_1) \tag{1}$$

where $y'|_{x=x_1}$ is the slope of the tangent to the curve evaluated at $x = x_1$.

We could fully expand the given expression relating x and y to prepare for differentiation, as:

$$3x^4 + 6x^2y^2 + y^4 = 100x^2 - y^2$$

This one is a bit complicated, and I think this expanded form will be slightly messier, i.e., offer opportunities for error, than just implicitly differentiating the equation in the given form, so let's just do that.

$$3(x^2 + y^2)^2 = 100(x^2 - y^2)$$

$$\frac{d}{dx}\left[3(x^2 + y^2)^2 = 100(x^2 - y^2)\right]$$

$$6(x^2 + y^2)(2x + 2yy') = 100(2x - 2yy') \qquad \text{(use the chain rule)}$$

$$12(x^2 + y^2)(x + yy') = 200(x - yy')$$

$$3(x^2+y^2)(x+yy') = 50(x-yy') \qquad \text{(simplify)}$$

Now, if this were a problem about finding *an expression* for y', we would have to do some algebra to isolate y'. However, this question is simply asking for the *value* of y' *at a specific point on the graph* of the given relationship. Note that the last step above contains only x's, y's, and y''s. It's at *this* point that we should substitute our given (x, y) pair, $(4, 2)$, into the expression:

$$\begin{aligned}
3(x^2+y^2)(x+yy') &= 50(x-yy') \\
3(4^2+2^2)(4+2y') &= 50(4-2y') && \text{(substitute)} \\
3\cdot 20(4+2y') &= 50(4-2y') \\
6(4+2y') &= 5(4-2y') \\
24+12y' &= 20-10y' \\
22y' &= -4 \\
y' &= -\frac{2}{11} && \text{(simplify)}
\end{aligned}$$

We are nearly done, as we have all we need: $x_1 = 4$, $y_1 = 2$, and $y'|_{x=4} = -\frac{2}{11}$. Substituting into (1), we have:

$$\begin{aligned}
y - y_1 &= y'|_{x=x_1}(x-x_1) \\
y - 2 &= -\frac{2}{11}(x-4) && \text{(substitute)} \\
y &= -\frac{2}{11}(x-4)+2 \\
y &= -\frac{2}{11}x+\frac{8}{11}+2 \\
y &= -\frac{2}{11}x+\frac{30}{11} && \text{(simplify)}
\end{aligned}$$

Having a look at `https://www.desmos.com/calculator/pstkeokx8m`, we see that this is indeed the correct equation for the tangent line at $(4, 2)$ on the given graph.

To summarize, this article is both about a challenging implicit differentiation problem, as well as keeping in mind what is being asked, in order to avoid doing more than what is needed. In particular, when we decided to not solve entirely for y', we avoided some tricky algebra, and thus, avoided opportunities to make errors.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

My email address is: phil.petrocelli@gmail.com.

Please visit `https://mymathteacheristerrible.com` for other study guides. Please tell others about it.

Please donate

I write these study guides with interest in good outcomes for math students and to be a part of the solution. If you would consider donating a few dollars to me so that these can remain free to everyone who wants them, please visit my PayPal and pay what you feel this is worth to you. Every little bit helps.

My PayPal URL is: `https://paypal.me/philpetrocelli`.

Thank you so much.