A tough limit problem, two ways

Phil Petrocelli, mymathteacheristerrible.com

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Not fun. Here's how we crank through it. Options.

Question.

Determine:

$$\lim_{x\to 2} \frac{\sqrt{2+x}-\sqrt{8-x^2}}{2-\sqrt{6-x}}$$

Solution.

Regular readers already know that rational functions are hard. This one looks ugly. For simplicity, let's substitute and show that this is not the way to go:

$$\lim_{x\to 2} \frac{\sqrt{2+x}-\sqrt{8-x^2}}{2-\sqrt{6-x}} = \frac{\sqrt{2+2}-\sqrt{8-2^2}}{2-\sqrt{6-2}}$$

$$= \frac{0}{0} \qquad \text{(indeterminate form!)}$$

<u>**Option 1:**</u> L'Hôpital's Rule is coming, but isn't here yet

When we get a result like $\frac{0}{0}$ for a limit, it is a signal that we need to do some algebra on the the function we are given. For rational functions, there are all manner of tricks for dealing with them.[1] The square roots in this example are a loud signal that we should use the conjugate trick on the fraction to massage the denominator, numerator, or both(!). Let's start with the conjugate of the denominator:

$$\lim_{x\to 2} \frac{\sqrt{2+x}-\sqrt{8-x^2}}{2-\sqrt{6-x}} = \lim_{x\to 2} \frac{\sqrt{2+x}-\sqrt{8-x^2}}{2-\sqrt{6-x}} \cdot \frac{2+\sqrt{6-x}}{2+\sqrt{6-x}} \qquad \text{(multiply all by conjugate of bottom)}$$

$$= \lim_{x\to 2} \frac{(\sqrt{2+x}-\sqrt{8-x^2})\cdot(2+\sqrt{6-x})}{4-(6-x)}$$

$$= \lim_{x\to 2} \frac{(\sqrt{2+x}-\sqrt{8-x^2})\cdot(2+\sqrt{6-x})}{4-6+x}$$

[1] Check out my article: `https://mymathteacheristerrible.com/blog/2019/2/24/free-calculus-study-guide-difficult-limits`

$$= \lim_{x\to 2} \frac{(\sqrt{2+x} - \sqrt{8-x^2}) \cdot (2+\sqrt{6-x})}{-2+x}$$

It appears that the conjugate trick with the denominator didn't work, as we still get $\frac{0}{0}$ when we substitute 2 for x. More work is needed. The next thing to try is to also multiply top and bottom by the conjugate of the *original numerator* (note that it's still there even after the denominator conjugate work) and see what happens:

$$\lim_{x\to 2} \frac{\sqrt{2+x} - \sqrt{8-x^2}}{2-\sqrt{6-x}} = \lim_{x\to 2} \frac{(\sqrt{2+x} - \sqrt{8-x^2}) \cdot (2+\sqrt{6-x})}{-2+x} \qquad \text{(start after first conj trick)}$$

$$= \lim_{x\to 2} \frac{(\sqrt{2+x} - \sqrt{8-x^2}) \cdot (2+\sqrt{6-x})}{-2+x} \cdot \frac{\sqrt{2+x} + \sqrt{8-x^2}}{\sqrt{2+x} + \sqrt{8-x^2}} \qquad \text{(mult by conjugate of top)}$$

$$= \lim_{x\to 2} \frac{(2+x) - (8-x^2) \cdot (2+\sqrt{6-x})}{(-2+x) \cdot (\sqrt{2+x} + \sqrt{8-x^2})}$$

$$= \lim_{x\to 2} \frac{(2+x-8+x^2) \cdot (2+\sqrt{6-x})}{(-2+x) \cdot (\sqrt{2+x} + \sqrt{8-x^2})}$$

$$= \lim_{x\to 2} \frac{(-6+x+x^2) \cdot (2+\sqrt{6-x})}{(-2+x) \cdot (\sqrt{2+x} + \sqrt{8-x^2})}$$

$$= \lim_{x\to 2} \frac{(-6+x+x^2)\cdot}{(-2+x)} \cdot \frac{(2+\sqrt{6-x})}{(\sqrt{2+x} + \sqrt{8-x^2})} \qquad \text{(split into 2 fractions)}$$

$$= \lim_{x\to 2} \frac{(x^2+x-6)\cdot}{(x-2)} \cdot \frac{(2+\sqrt{6-x})}{(\sqrt{2+x} + \sqrt{8-x^2})} \qquad \text{(rearr polynomial factors)}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+3)\cdot}{(x-2)} \cdot \frac{(2+\sqrt{6-x})}{(\sqrt{2+x} + \sqrt{8-x^2})} \qquad \text{(factor)}$$

$$= \lim_{x\to 2} (x+3) \cdot \frac{(2+\sqrt{6-x})}{(\sqrt{2+x} + \sqrt{8-x^2})} \qquad \text{(cancel the } (x-2) \text{ factors)}$$

Ok, *now* let's have a look at substituting:

$$\lim_{x\to 2} \frac{\sqrt{2+x} - \sqrt{8-x^2}}{2-\sqrt{6-x}} = \lim_{x\to 2} (x+3) \cdot \frac{(2+\sqrt{6-x})}{(\sqrt{2+x} + \sqrt{8-x^2})}$$

$$= (2+3) \cdot \frac{(2+\sqrt{6-2})}{(\sqrt{2+2} + \sqrt{8-2^2})}$$

$$= 5 \cdot \frac{(2+\sqrt{4})}{(\sqrt{4} + \sqrt{4})}$$

$$= 5 \cdot \frac{4}{4}$$

$$= 5$$

5 is the correct answer.

Option 2: L'Hôpital's Rule

The basic requirement for using L'Hôpital's Rule is that we get an indeterminate form when substituting. This is certainly true of this limit, so it *is* allowable. Let's have a look.

$$\lim_{x\to 2} \frac{\sqrt{2+x}-\sqrt{8-x^2}}{2-\sqrt{6-x}}$$

is in the form $\frac{f(x)}{g(x)}$, where $f(x) = \sqrt{2+x} - \sqrt{8-x^2}$ and $g(x) = 2 - \sqrt{6-x}$, and $f(2) = 0$ and $g(2) = 0$. Let's take some derivatives:

$$\begin{aligned} f(x) &= \sqrt{2+x} - \sqrt{8-x^2} \\ &= (2+x)^{\frac{1}{2}} - (8-x^2)^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(2+x)^{-\frac{1}{2}} + x(8-x^2)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} g(x) &= 2 - \sqrt{6-x} \\ &= 2 - (6-x)^{\frac{1}{2}} \\ g'(x) &= \frac{1}{2}(6-x)^{-\frac{1}{2}} \end{aligned}$$

Applying L'Hôpital's Rule, we can rephrase our limit as:

$$\begin{aligned} \lim_{x\to 2} \frac{\sqrt{2+x}-\sqrt{8-x^2}}{2-\sqrt{6-x}} &= \lim_{x\to 2} \frac{\frac{1}{2}(2+x)^{-\frac{1}{2}} + x(8-x^2)^{-\frac{1}{2}}}{\frac{1}{2}(6-x)^{-\frac{1}{2}}} && \text{(replace top and bottom with their derivatives)} \\ &= \lim_{x\to 2} \frac{(2+x)^{-\frac{1}{2}} + 2x(8-x^2)^{-\frac{1}{2}}}{(6-x)^{-\frac{1}{2}}} && \text{(factor and cancel)} \end{aligned}$$

Rather than do more algebra (and possibly introduce errors), let's substitute right away, as-is:

$$\begin{aligned} \lim_{x\to 2} \frac{(2+x)^{-\frac{1}{2}} + 2x(8-x^2)^{-\frac{1}{2}}}{(6-x)^{-\frac{1}{2}}} &= \frac{(2+2)^{-\frac{1}{2}} + 2\cdot 2(8-2^2)^{-\frac{1}{2}}}{(6-2)^{-\frac{1}{2}}} \\ &= \frac{(4)^{-\frac{1}{2}} + 4(8-4)^{-\frac{1}{2}}}{(4)^{-\frac{1}{2}}} \\ &= \frac{(4)^{-\frac{1}{2}} + 4(4)^{-\frac{1}{2}}}{(4)^{-\frac{1}{2}}} \end{aligned}$$

$$= 1 + 4$$
$$= 5$$

Some points about both approaches

1. By the time you get to this kind of problem in Calculus, you likely have recognized that **algebra and algebra trickery are both essential and kind of frightening**, at times. Agreed.

2. In both examples, note my **decisions to *not* expand things beyond what is essential** to work on. Specifically, leaving things in factored forms, especially for functions where there are domain concerns (logs, even roots, rational functions, etc.). It's a whole lot easier to see where things go to zero or have a sign change, etc. when they are in factored form.

   As an example, recall your Algebra 2: can you tell where a polynomial goes to zero when it is in standard form? Right - more work is needed. How about when it is in factored form? Absolutely. Can pick them right out of the factors.

3. **Organization is key.**[2]

   In the conjugate solution, I did things like rearrange polynomials into their standard form representations, for instance. Makes things clearer and more familiar, I think. I also took care to separate out parts of fractions into products of fractions: one fraction we were working on, the others just hanging out until they needed to be worked.

   In the L'Hôpital's Rule solution, I did some "side work" of separating numerator and denominator into $f(x)$ and $g(x)$, in order to make the derivative work easier, clearer. These little things can make all the difference sometimes.

4. **Staying focused on what the problem is really about.**

   In the conjugate solution, the essentials for this particular rational function were about using the conjugate trick to get rid of some square roots - note that everything else was left in factored form, even down to the point where we were ready for substitution.

   In the L'Hôpital's Rule solution, any simplifying is potential for introducing error. Look how that one ultimately turned out. Do we really care that negative exponents suggest that we should move things to the numerator or denominator? Absolutely not. The cancellation step near the end justifies this type of thinking. In addition, this is a great time-saving technique for time-intensive exams.

Done.

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[2] A great example of this:
https://mymathteacheristerrible.com/blog/2019/7/19/when-managing-the-path-to-solution-is-really-important-a-tricky-3rd-de

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

My email address is: phil.petrocelli@gmail.com.

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