Converting a repeating decimal to a fraction: some techniques

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Repeating decimals are rational numbers. Rational numbers can be expressed as ratios of integers. Let's look at some techniques for moving easily between repeating decimals and their ratio forms.

Question.

How can we convert between a number's repeating decimal form and its ratio form?

Discussion.

1 The interesting pattern of dividing by 9

Besides $0.\overline{0}$, the most interesting place to start is the very basic $0.\overline{1}$. It makes sense to choose this value because, under multiplication and division, 1 is the identity property, i.e., anything multiplied by 1 is itself, as is anything divided by 1. We can make use of this idea in a few ways.

We know that the repeating decimal $0.\overline{1}$'s ratio form is $\frac{1}{9}$. We can use this simple fact to examine all fractions - all *ninths* - between zero and 1:

$$\frac{0}{9} = 0$$

$$\frac{1}{9} = 0.\overline{1}$$

$$\frac{2}{9} = 2 \cdot \frac{1}{9} = 2 \cdot 0.\overline{1} = 0.\overline{2}$$

$$\frac{3}{9} = 3 \cdot \frac{1}{9} = 3 \cdot 0.\overline{1} = 0.\overline{3}$$

$$\frac{4}{9} = 4 \cdot \frac{1}{9} = 4 \cdot 0.\overline{1} = 0.\overline{4}$$

$$\frac{5}{9} = 5 \cdot \frac{1}{9} = 5 \cdot 0.\overline{1} = 0.\overline{5}$$

$$\frac{6}{9} = 6 \cdot \frac{1}{9} = 6 \cdot 0.\overline{1} = 0.\overline{6}$$

$$\frac{7}{9} = 7 \cdot \frac{1}{9} = 7 \cdot 0.\overline{1} = 0.\overline{7}$$

$$\frac{8}{9} = 8 \cdot \frac{1}{9} = 8 \cdot 0.\overline{1} = 0.\overline{8}$$

$$\frac{9}{9} = 1$$

Note the situations for $\frac{3}{9}$ and $\frac{6}{9}$, specifically, think of them in their reduced forms - $\frac{1}{3}$ and $\frac{2}{3}$ - respectively. Those are repeating decimals you likely already know pretty well.

That's right: whatever the number in the numerator for any ninth between 0 and 1, that number forms the repeating portion - 2 digits - of the decimal form. This also works for improper fractions and mixed numbers involving ninths:

$$\frac{37}{9} = \frac{36}{9} + \frac{1}{9} = 4 + \frac{1}{9} = 4.\overline{1} \qquad \text{(break up into something divisible by 9 plus a ninth between 0 and 1)}$$

$$4\frac{8}{9} = 4 + \frac{8}{9} = 4 + 0.\overline{8} = 4.\overline{8} \qquad \text{(already divided up into integer and fraction - simple!)}$$

2 The interesting pattern of dividing by 99

Following the same logic as the previous section, $\frac{1}{99} = 0.\overline{01}$. There are 100 99^{ths} between zero and 1. Here are just a few:

$$\frac{37}{99} = 37 \cdot \frac{1}{99} = 37 \cdot 0.\overline{01} = 0.\overline{37}$$

$$\frac{98}{99} = 98 \cdot \frac{1}{99} = 98 \cdot 0.\overline{01} = 0.\overline{98}$$

$$\frac{54}{99} = 54 \cdot \frac{1}{99} = 54 \cdot 0.\overline{01} = 0.\overline{54}$$

And just like the previous section on ninths, when the denominator is 99, i.e., *two nines*, the numerator, if between 1 and 98, forms the repeating portion of the decimal form. Also works for improper fractions, as mentioned above.

3 You guessed it: the pattern for dividing by 999

Briefly, a pattern similar to that of 9^{ths} and 99^{ths} holds. For any fraction between $\frac{1}{999}$ and $\frac{998}{999}$, the numerator forms the basis for a 3-digit repeating pattern in decimal form. Like:

$$\frac{307}{999} = 307 \cdot \frac{1}{999} = 307 \cdot 0.\overline{001} = 0.\overline{307}$$

$$\frac{989}{999} = 989 \cdot \frac{1}{999} = 989 \cdot 0.\overline{001} = 0.\overline{989}$$

$$\frac{54}{99} = 54 \cdot \frac{1}{99} = 54 \cdot 0.\overline{01} = 0.\overline{54}$$

4 Codifying the pattern

I'm terrible at figuring out patterns sometimes, to be honest. Fortunately, `https://en.wikipedia.org/wiki/Repeating_decimal` exists. To express any n-digit number that consists of all 9's - 9...9 - we can use the simple expression, $10^n - 1$, where n is the number of digits. Brilliant!

We can simply say:

Theorem 1. *Any rational number with a denominator of $10^n - 1$ produces a repeating decimal whose repeating part consists of n digits.*

The converse is also true:

Theorem 2. *A repeating decimal whose repeating part consists of n digits can be expressed as a rational number with a denominator of $10^n - 1$.*

Examples (read the equations/comments forwards and backwards to cover theorem and converse):

$$0.\overline{234} = \frac{234}{999} \qquad \text{(has 3 digits in its repeating part, denominator is 999)}$$

$$0.\overline{666777} = \frac{666777}{999999} \qquad \text{(has 6 digits in its repeating part, denominator is 999999)}$$

Take a moment to consider, that, even if we are imprecise about the the repeating digits of a decimal, we can still get the correct answer:

$$0.\overline{4} = 0.\overline{44} = 0.\overline{444} = 0.\overline{4444}$$

In their fraction forms, we have:

$$\frac{4}{9} = \frac{44}{99} = \frac{444}{999} = \frac{4444}{9999}$$

Note that all of those fractions reduce to $\frac{4}{9}$.[1] The reader should try, say: $0.\overline{1002}$, $0.\overline{10021002}$, and $0.\overline{100210021002}$.

5 Techniques for converting between repeating decimal and ratio forms

This is by no means an exhaustive discussion of techniques, rather, here are a few things to consider as you work with fractions and decimals.

5.1 Decimals with repeating parts that start at the decimal point

To summarize the technique we've been using so far here, if a repeating decimal starts at the decimal point and consists of n repeating digits, its fraction form is:

$$\frac{\text{repeating digits}}{10^n - 1}$$

Review the examples above.

5.2 Decimals with repeating parts that don't start at the decimal point

It is the case that nature, and the numbers we use to represent natural things, are not so convenient. No matter, as there are some techniques that we can use to determine the rational form of a repeating decimal, in general.

5.2.1 Dividing up the number into relevant constituent parts

Let's have a look at $234.78\overline{1}$, for instance. The leading .78 in the decimal part is not repeating, and it comes between the decimal point and the repeating 1. Let's see how we might deal with this. Consider:

$$234.78\overline{1} = 234 + 0.78 + 0.00\overline{1} \qquad \text{(break up into constituent parts)}$$

[1]Now seems like a great time to mention `https://mymathteacheristerrible.com/blog/2019/12/12/using-the-fraction-features-of-the-ti-84-family-of-calculators`, if you own a TI-84 calculator. Best fraction reducer ever!

$$= 234 + \frac{78}{100} + 0.00\overline{1} \qquad \text{(we know 0.78 as a ratio)}$$

$$= 234 + \frac{78}{100} + \frac{0.\overline{1}}{100} \qquad \text{(Trick! That is, that } 0.\overline{1} \text{ moved to the right 2 digits is } \textit{dividing by 100}\text{)}$$

$$= 234 + \frac{78}{100} + \frac{1}{9} \cdot \frac{1}{100} \qquad \text{(clever use of } 0.\overline{1} = \frac{1}{9}\text{)}$$

$$= 234 + \frac{78}{100} + \frac{1}{900} \qquad \text{(expand)}$$

$$= 234 + \frac{703}{900} \qquad \text{(add fractions and reduce)}$$

$$= 234\frac{703}{900} \qquad \text{(mixed number form)}$$

$$= \frac{211303}{900} \qquad \text{(improper fraction form)}$$

Maybe the most difficult part of this example was to multiply 78 by 9 when adding the two fractions? Super simple. Really interesting technique.

Also, note that we can take advantage of Theorem 1. If there is a 1-digit repeating part, we can be sure that ninths are involved *somehow*. This is the case, no matter where the repeating part is.

5.2.2 A method where $(10^n - 1)$ shows up during solution

Let's look at $234.78\overline{1}$ again. This trick involves getting *one* digit of the repeating part into the whole number part, as well as moving the repeating part to right after the decimal point. This means we should multiply our original number by 1000, or 10^3:

$$\begin{aligned}
x &= 234.78\overline{1} \\
10^2 \cdot x &= 23478.\overline{1} && \text{(move repeating part to be after decimal)} \\
10^3 \cdot x &= 234781.\overline{1} && \text{(move one repeating sequence into whole number part)} \\
10^3x - 10^2x &= 234781 - 23478 && \text{(subtract - this clears the repeating decimal)} \\
10^2x(10^1 - 1) &= 211303 && \text{(factor - note how the 1 sneaks into } 10^1 \text{ indicating one repeating digit)} \\
10^2x(9) &= 211303 && \text{(simplify)} \\
900x &= 211303 && \text{(multiply)} \\
x &= \frac{211303}{900}
\end{aligned}$$

5.2.3 Repeating parts where there are *k* leading zeroes

This shows up in (5.2.1), above, although we have to dig for it a little. Let's just consider the fraction part of the number we worked with: $0.78\overline{1}$:

$$0.78\overline{1} = 0.78 + 0.00\overline{1}$$

Looking more closely at $0.00\overline{1}$, note that n (the number of repeating digits) is 1. Let's call k the number of zeroes in front of the repeating digits; so, $k = 2$. Note:

$$\begin{aligned}
0.00\overline{1} &= 1 \cdot \frac{1}{900} && \text{(1 is the part that repeats)} \\
&= 1 \cdot \frac{1}{9} \cdot \frac{1}{100} \\
&= 1 \cdot \frac{1}{10^1 - 1} \cdot \frac{1}{10^2} \\
&= \text{(repeating digits)} \cdot \frac{1}{10^n - 1} \cdot \frac{1}{10^k}
\end{aligned}$$

This formula works for more complicated decimals, too:

$$\begin{aligned}
0.000\overline{4567} &= 4567 \cdot \frac{1}{10^4 - 1} \cdot \frac{1}{10^3} && \text{(4567 is the part that repeats)} \\
&= 4567 \cdot \frac{1}{10000 - 1} \cdot \frac{1}{1000} \\
&= 4567 \cdot \frac{1}{9999} \cdot \frac{1}{1000} \\
&= 4567 \cdot \frac{1}{9999000} \\
&= \frac{4567}{9999000}
\end{aligned}$$

There are probably more techniques that I will discover. I will update this article when I find more!

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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