Calculus thinking: Seeing and applying a cool algebra trick!

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This is an interesting derivatives question that can be dealt with more easily if we employ a (somewhat) advanced, possibly non-obvious algebra trick.

Question.

Find the the turning points of:

$$f(x) = x\sqrt{64 - x^2}$$

Solution.

1 Calculating the derivative

This seems relatively straightforward for even those who are new to derivatives. The traditional way to go about this is to take the first derivative, then solve for its zeroes. Let's do just that.

1.1 First, the "traditional" way

Considering the first derivative, we see it is going to be a product rule (the x times *something*) and the chain rule (the square root term):

$$f(x) = x\sqrt{64 - x^2}$$

$$f(x) = x(64 - x^2)^{\frac{1}{2}} \qquad \text{(rewrite in exponent form)}$$

$$\begin{aligned} f'(x) &= 1 \cdot (64 - x^2)^{\frac{1}{2}} \ + \ x \cdot \frac{1}{2}(64 - x^2)^{-\frac{1}{2}}(-2x) && \text{(product rule, one term requires chain rule)} \quad (1) \\ &= (64 - x^2)^{\frac{1}{2}} \ - \ x^2(64 - x^2)^{-\frac{1}{2}} && \text{(simplify, clean up)} \\ &= (64 - x^2)^{-\frac{1}{2}}\left[(64 - x^2) - x^2\right] && \text{(factor)} \\ &= (64 - x^2)^{-\frac{1}{2}}(64 - 2x^2) && \text{(simplify)} \end{aligned}$$

$$f'(x) = \frac{64 - 2x^2}{\sqrt{64 - x^2}} \qquad \text{(rewrite as a fraction, done.)} \qquad (2)$$

Not too bad. I feel like it does get a bit messy with all of the terms involved. That happens, like, right away. We can see it comes mostly from dealing with the product rule in (1). Let's see how we might make it better using **calculus thinking**.

1.2 Now, let's use calculus thinking

For reference, please have a look at my article on **calculus thinking**[1] and consider this application.

Is there a way that we might *avoid* using the product rule by some clever trick? Turns out, we can. The trick lies in the rules of exponents. The x hanging out in front of the radical in our original definition of $f(x)$ is the thing that makes the product rule a requirement:

$$f(x) = x\sqrt{64 - x^2}$$

It sure would be nice if we could do something about that, right? We can, if we *un*simplify the expression by saying:

$$x = \sqrt{x^2}$$

and substituting, making our definition of $f(x)$:

$$f(x) = \sqrt{x^2}\sqrt{64 - x^2}$$

Now, using the rules of roots and exponents, we can combine both terms under *one* radical, as:

$$f(x) = \sqrt{x^2(64 - x^2)} \tag{3}$$

Looking ahead a bit, we can see that *this* expression will require *only* the chain rule. Here's how:

$$\begin{aligned}
f(x) &= \sqrt{x^2(64 - x^2)} \\
&= \sqrt{64x^2 - x^4} && \text{(expand under the radical, giving a simple polynomial)} \\
&= (64x^2 - x^4)^{\frac{1}{2}} && \text{(rewrite radical in exponent form)}
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \frac{1}{2}(64x^2 - x^4)^{-\frac{1}{2}} \cdot (128x - 4x^3) && \text{(apply chain rule)} \\
&= \frac{1}{2}(64x^2 - x^4)^{-\frac{1}{2}} \cdot 2x(64 - 2x^2) && \text{(factor second term)} \\
&= (64x^2 - x^4)^{-\frac{1}{2}} \cdot x(64 - 2x^2) && \text{(simplify)}
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \frac{x(64 - 2x^2)}{\sqrt{64x^2 - x^4}} && \text{(rewrite in fraction form)} \\
&= \frac{x(64 - 2x^2)}{\sqrt{x^2(64 - x^2)}} && \text{(factor denominator under radical)} \qquad (4)
\end{aligned}$$

Note that the denominator of (4) is the same as (3)! Continuing with a substitution:

$$\begin{aligned}
f'(x) &= \frac{x(64 - 2x^2)}{\sqrt{x^2(64 - x^2)}} && \text{(factor denominator under radical)} \\
&= \frac{x(64 - 2x^2)}{x\sqrt{64 - x^2}} && \text{(substitute in the denominator: original } f(x)\text{)} \\
f'(x) &= \frac{64 - 2x^2}{\sqrt{64 - x^2}} && \text{(simplify)} \qquad (5)
\end{aligned}$$

Finally, we see that (5) is the same as (2), no product rule needed.

[1] https://mymathteacheristerrible.com/blog/2019/5/15/calculus-thinking-a-definition-i-use-over-and-over-with-my-students

2 Finding the turning points

The remainder of this problem is fairly easy from here on out. Let's start with the graph of this situation. If you head to `https://www.desmos.com/calculator/hiahkpsvte`, we can see what we already know about this function's domain: $f(x)$ is defined only on $x \epsilon [-8, 8]$. Having a look at the derivative stated in (5), we see that the domain of $f'(x)$ has a more restrictive domain than that of $f(x)$, as the denominator cannot be zero. That being said, the domain of $f'(x)$ is a subset of the domain of $f(x)$, namely, $x \epsilon (-8, 8)$.[2] To calculate what we see in the graph as the two turning points of $f(x)$:

$$\begin{aligned}
f'(x) &= 0 \\
64 - 2x^2 &= 0 && \text{(consider only the numerator)} \\
2x^2 &= 64 \\
x &= \pm\sqrt{32} \\
x &= \pm 4\sqrt{2}
\end{aligned}$$

Making our turning points:

$$\begin{aligned}
f(4\sqrt{2}) &= 4\sqrt{2}\sqrt{64 - (4\sqrt{2})^2} \\
&= 4\sqrt{2}\sqrt{64 - 16 \cdot 2} \\
&= 4\sqrt{2}\sqrt{64 - 32} \\
&= 4\sqrt{2}\sqrt{32} \\
&= 4\sqrt{2} \cdot 4\sqrt{2} \\
f(4\sqrt{2}) &= 32
\end{aligned}$$

And, similarly, $f(-4\sqrt{2}) = -32$.

So, our two turning points are $(4\sqrt{2}, 32)$ and $(-4\sqrt{2}, -32)$.

Done.

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[2] the square brackets just changed to parentheses! Subtle!

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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