A challenging trig limit with cool algebra tricks!

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Sometimes a Calculus problem is about non-obvious algebra. Here is one such instance.

Question.

Calculate:

$$L = \lim_{x \to 0} \frac{sin^2(\frac{x}{2})}{x^2}$$

Solution.

This is a weird one. It *sort-of* looks like one that should be easy, but it is fraught with issues, when it comes down to the algebra. If you're brand new Calculus student, the method available in section **2** is *not* in your toolbox yet. Also not fun.

This really can be thought of as a Calculus part (the limit itself) and an algebra part (rephrasing that fraction so that it's more useful)[1].

1 Doing the algebra: The trick!

the limit we're working on *sort of* looks like a basic trigonometric limit that we all need to memorize:

$$\lim_{x \to 0} \frac{sin\ x}{x} = 1 \tag{1}$$

This limit makes use of the *small angle approximation for sine*. In more detailed language, we can say that *for small angles, the sine of the angle is approximately equal to the angle itself.* Note that a crucial detail is that the angle in the numerator matches the term in the denominator. This will guide our algebra.

First thing to notice is that the fraction has a squared term in the numerator and denominator. But the *angle* is not squared in the numerator, as the variable in the denominator is. The angle is a bit "trapped" in the $\sin^2(\frac{x}{2})$ term. We can, however, make use of this rule of exponents, related to fractions:

$$\frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2$$

Applying this to our limit, we have:

[1] I call this **Calculus thinking**, click for more discussion

$$L = \lim_{x \to 0} \frac{sin^2(\frac{x}{2})}{x^2}$$

$$= \lim_{x \to 0} \left(\frac{sin\frac{x}{2}}{x} \right)^2 \qquad \text{(apply exponent rule)}$$

This one move clarifies things, doesn't it? Especially the situation with the $\frac{x}{2}$ in the numerator and the x in the denominator - they currently do not match, as needed by (1), above. Let's use an algebra trick. We can say

$$x = \frac{2x}{2}$$

This is still equal to x, as we simply multiplied by 1. Substitute:

$$L = \lim_{x \to 0} \left(\frac{sin\frac{x}{2}}{x} \right)^2$$

$$= \lim_{x \to 0} \left(\frac{sin\frac{x}{2}}{\frac{2x}{2}} \right)^2 \qquad \text{(new denominator)}$$

One more slick move. We can factor the 2 from the numerator of $\frac{2x}{2}$ and factor the denominator of the full fraction:

$$L = \lim_{x \to 0} \left(\frac{sin\frac{x}{2}}{\frac{2x}{2}} \right)^2$$

$$= \lim_{x \to 0} \left(\frac{sin\frac{x}{2}}{2 \cdot \frac{x}{2}} \right)^2 \qquad \text{(factor the denominator)}$$

$$= \lim_{x \to 0} \left(\frac{1}{2} \cdot \frac{sin\frac{x}{2}}{\frac{x}{2}} \right)^2 \qquad \text{(factor the full fraction!)}$$

Check out what's happening! Let's continue, using one more rule of exponents:

$$L = \lim_{x \to 0} \left(\frac{1}{2} \cdot \frac{sin\frac{x}{2}}{\frac{x}{2}} \right)^2$$

$$= \lim_{x \to 0} \left[\left(\frac{1}{2} \right)^2 \cdot \left(\frac{sin\frac{x}{2}}{\frac{x}{2}} \right)^2 \right] \qquad \text{(distribute the exponent to both terms)}$$

$$= \left(\frac{1}{2} \right)^2 \cdot \lim_{x \to 0} \left(\frac{sin\frac{x}{2}}{\frac{x}{2}} \right)^2 \qquad \text{(property of limits, factor out constant)}$$

$$= \frac{1}{4} \cdot \lim_{x \to 0} \left(\frac{sin\frac{x}{2}}{\frac{x}{2}} \right)^2 \qquad \text{(simplify)}$$

This puts us in a really good spot. There is a rule of limits that we can use to finish this one completely. The rule is about the limit of a composite function, something of the form $f(g(x))$, which is what we have here. Our outer function is $f(x) = x^2$, and the rule says, that, so long as the outer function is continuous, we can "invite" the limit operator inside the *outer* function, so that the limit operator is applied to the *inner* function. Let's do this in our next step.

$$L = \frac{1}{4} \cdot \lim_{x \to 0} \left(\frac{sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

$$= \frac{1}{4} \cdot \left(\lim_{x \to 0} \frac{sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \qquad \text{(move the limit operator inside)}$$

Now for the payoff! If we examine what's inside the parentheses, it's our basic trig limit from (1)! Since the angle and the term in the denominator are the same, this limit evaluates to 1, and we have:

$$L = \frac{1}{4} \cdot \left(\lim_{x \to 0} \frac{sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

$$= \frac{1}{4} \cdot 1^2 \qquad \text{(calculate the limit)}$$

$$L = \frac{1}{4} \qquad \text{(simplify)}$$

Super cool, yes?

2 L'Hôpital's Rule is possible, but it's hard to manage, IMO

As a reminder, L'Hôpital's Rule states that, when evaluating a limit of the form

$$L = \lim_{x \to a} \frac{f(x)}{g(x)}$$

results in an indeterminate form, such as $\frac{0}{0}$, can reevaluate the limit as

$$L = \lim_{x \to a} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)}$$

with consecutive applications of this rule until we rid ourselves of the indeterminate form in our result.

We *can* use this rule with our limit:

$$L = \lim_{x \to 0} \frac{sin^2(\frac{x}{2})}{x^2}$$

$$= \frac{0}{0} \qquad \text{(indeterminate form)}$$

$$= \lim_{x \to 0} \frac{sin(\frac{x}{2}) \cdot cos(\frac{x}{2})}{2x} \qquad \text{(apply L'H, differentiate top and bottom)} \qquad (2)$$

$$= \frac{0}{0} \qquad \text{(indeterminate form - AGAIN!)}$$

We *can* apply L'Hôpital's Rule again, but the astute Calculus student will recognize that differentiating the numerator now involves the use of the product rule for differentiation, or a slick algebra trick like using the product rule for limits (*the limit of a product is the product of the limits*), where we split apart the fraction and deal with it that way, possibly, but even as I type this, I see that peeling off the $cos(\frac{x}{2})$ term will still necessitate use of L'Hôpital's Rule on the fraction with the $sin(\frac{x}{2})$ term. This is why I prefer the method from section 1. For those interested, I'll show the two options we have at this point, starting with (2).

2.1 Straight L'Hôpital's Rule again

$$L = \lim_{x\to\ 0} \frac{sin(\frac{x}{2}) \cdot cos(\frac{x}{2})}{2x}$$

$$= \lim_{x\to\ 0} \frac{\frac{1}{2}cos^2(\frac{x}{2}) - \frac{1}{2}sin^2(\frac{x}{2})}{2} \qquad \text{(differentiate, apply product rule in numerator)}$$

$$= \lim_{x\to\ 0} \frac{cos\ x}{4} \qquad \text{(simplify, using double angle identity for cosine)}$$

$$L = \frac{1}{4} \qquad \text{(evaluate the limit)}$$

Interesting result, with the appearance of the double angle identity for cosine. The reason that I don't recommend this is not because it doesn't work, rather, for some, using the product rule in the numerator might increase the chance of introducing an error.

2.2 Limit of a product

$$L = \lim_{x\to\ 0} \frac{sin(\frac{x}{2}) \cdot cos(\frac{x}{2})}{2x}$$

$$= \lim_{x\to\ 0} \frac{sin(\frac{x}{2})}{2x} \cdot \lim_{x\to\ 0} cos\left(\frac{x}{2}\right) \qquad \text{(split up, using product property of limits)}$$

$$= \frac{1}{2} \cdot \lim_{x\to\ 0} \frac{sin(\frac{x}{2})}{x} \cdot 1 \qquad \text{(evaluate, factor, simplify)}$$

$$= \frac{1}{2} \cdot \lim_{x\to\ 0} \frac{sin(\frac{x}{2})}{x}$$

At this point, we can apply L'Hôpital's Rule to this limit, or, we can use our algebra trick, from Section 1, above. More specifically, we used L'Hôpital's Rule to possibly avoid some algebra, yet, as it simplifies, the problem still appears to be demanding it. Maybe. It appears that we might *not* be better off, really, at this point.

I guess it's a matter of how one sees these kinds of things, which is mostly about personal style of solving problems like this. This whole article, though opinionated at times, validates that, somewhat.

Done.

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I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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