Infamous question from an MIT entrance exam, 1876

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To me, this problem was more about making choices that ease solution by abstracting away some details we might not know what to do with, so as not to derail us as we work. This is my take, mostly an application of function notation.

Question.

A father said to his son:
Two years ago I was three times as old as you, but in fourteen years I shall be only twice as old as you. What were the ages of each?[1]

Solution.

1 First impressions

There is a great solution on Presh Talwaker's excellent Mind your Decisions YouTube channel - please have a look!

When I began solving, I did not look at any hints nor solutions. I what I did know (and this will play out later) is that the passage of time is linear, that is, graphically speaking, a line with a slope of 1: 1 year passes (run), age increases by 1 (rise). I was not, however, sure about how to account for that in my solution, at first. Did I even need to? And, more specifically, we look to be talking about some $y-$coordinate being n times more than another one (n, an integer), and how should that be dealt with?

Followers of my blog are probably familiar with my *push through it* approach to solving some problems that appear to have a myriad of details to manage. In this problem, we appeared to have too many variables. Doing something slick and then pushing it through was what I chose to do.

2 Choices, setup

The wording here is tricky. There are 3 phrases that needed to be dealt with:

- two years ago, meaning: *two years **before** some year*
- in fourteen years, meaning: *14 years **after** some year*
- an implied *now* or ***this** year*

[1] By way of Popular Mechanics:
`https://www.popularmechanics.com/science/math/a35206432/math-puzzle-mit-entrance-exam-viral-problem/`

Honestly, I didn't want to deal with them right away, so I found a way not to, temporarily, with an application of function notation. I used function notation to abstract away (hide) the bits that seemed *maybe* relevant at *some* point, but not *immediately* relevant? I pushed through. I let a function: $d(y)$ be the dad's age in year y and a function: $s(y)$ be the son's age in year y. Note that we only need one way to track a year, because the passage of years is the same for dad and son. And, most importantly, $s(y)$ and $d(y)$ can be thought of as the son's and dad's ages, respectively, in the current year. To flesh this out, relative to the problem, we actually know quite a lot, and we can note it all in terms of these two functions:

$$s(y) \qquad \text{(son's age in current year)}$$
$$d(y) \qquad \text{(dad's age in current year)}$$

$$s(y-2) \qquad \text{(son's age 2 years ago)}$$
$$d(y-2) \qquad \text{(dad's age 2 years ago)}$$

$$s(y+14) \qquad \text{(son's age 14 years after current year)}$$
$$d(y+14) \qquad \text{(dad's age 14 years after current year)}$$

Now, we can begin writing some equations having made these simple choices. Notice that "now", "2 years ago", and "in 14 years" are all handled quite well for the moment; they have their places, and we are ready to see how that might play out when we push through. Our equations turn out to be:

$$d(y-2) = 3 \cdot s(y-2) \qquad \text{(2 years ago, dad's age equalled 3 times son's age)} \tag{1}$$
$$d(y+14) = 2 \cdot s(y+14) \qquad \text{(14 years from now, dad's age will equal 2 times son's age)} \tag{2}$$

Yes, the $y-2$ terms and the $y-14$ terms have to be the same for son and dad, as we are talking about the relationships between their ages both 2 years ago and 14 years from now. An alternate way that we can read those terms is that they represent the same year.

3 Insights, a trick, solving it

I agree that (1) and (2) may not appear to be very helpful. They look like they would be a system of equations - maybe - except that the terms involving $y-2$ and $y+14$ are actually *not* like terms. It still seems like a bunch of information is missing. Like terms are needed!

Having a closer look, there is actually a key bit hiding in plain sight. If we take a closer look at the $y-2$ and $y+14$ terms, it turns out that there *is* a relationship between them. Note that we can calculate the number of years between $y-2$ years and $y+14$ years. 16 years. And, the way this relationship is used produces something supercool when we push it through (1) and (2). Notice, that, for any y, $s(y+16)$ is the same as $s(y)+16$. That is, the son's age 16 years from now is the same as the son's age now, plus 16 years. Same for dad. Using our notation, we have:

$$d(y+16) = d(y) + 16 \tag{3}$$
$$s(y+16) = s(y) + 16 \tag{4}$$

If we take $y-2$ to be our base year, we can use (3) and (4) by taking the all of the y's and replacing them with $y-2$, using the rules of function notation, giving:

$$d(y-2+16) = d(y-2) + 16$$
$$d(y+14) = d(y-2) + 16 \tag{5}$$

$$s(y-2+16) = s(y-2) + 16$$

$$s(y+14) = s(y-2) + 16 \tag{6}$$

Going back to (2), we can rewrite in terms of $s(y-2)$ and $d(y-2)$, making our equations:

$$d(y-2) = 3 \cdot s(y-2) \qquad \text{(equation (1), no change)} \tag{7}$$

$$\begin{aligned}
d(y+14) &= 2 \cdot s(y+14) && \text{(equation (2))} \\
d(y-2) + 16 &= 2\big[s(y-2) + 16\big] && \text{(substitute, making use of (5) \& (6))} \\
d(y-2) + 16 &= 2 \cdot s(y-2) + 32 && \text{(distribute)} \\
d(y-2) &= 2 \cdot s(y-2) + 16 && \text{(subtract 16 from both sides)}
\end{aligned} \tag{8}$$

Note that this reduces the number of terms in our equations, and now it looks like we have some like terms:

$$d(y-2) = 3 \cdot s(y-2) \tag{9}$$

$$d(y-2) = 2 \cdot s(y-2) + 16 \tag{10}$$

Now, let's use a clarifying technique to clean up our notation (abstract away the nonessentials) and to let us focus more on solving our system of linear equations. Let $u = d(y-2)$ and $v = s(y-2)$, giving:

$$\begin{aligned}
u &= 3v \\
u &= 2v + 16
\end{aligned}$$

So:

$$\begin{aligned}
3v &= 2v + 16 && \text{(substitution, set equal to each other)} \\
v &= 16 && \text{(solve for } v\text{)}
\end{aligned} \tag{11}$$

and:

$$\begin{aligned}
u &= 3v \\
u &= 3 \cdot 16 && \text{(substitute)} \\
u &= 48 && \text{(simplify)}
\end{aligned} \tag{12}$$

Going back to our uses of u and v, we see that both represent the dad's and son's ages, respectively, *2 years ago*. So, our ages for each - *today*, year y - are dad: $u + 2 = 48 + 2 =$ **50 years old**, son: $v + 2 = 16 + 2 =$ **18 years old**.

Note that we did not use anything involving y or $y-2$ or $y+14$ until the very end of our calculations. This is the abstraction that I set out to do early on: I chose not to worry about the details of how the years were calculated, because my intuition told me they were not that important, and that they could probably be "hidden" until they were absolutely needed. The real meat of the solution to this problem, to me, is when we created the system of equations by switching to u and v. Note that the system in u and v is not concerned with y at all; y is safely behind the scenes until we needed to calculate the actual ages, right at the very end, when we undid our switch to u and v.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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