L'Hôpital's Rule and Fundamental Theorem of Calculus (FTC) lurk!

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This is a super interesting question that I saw on Reddit (r/Calculus). Check this out!

Question.

Calculate:

$$\lim_{x\to 1} \frac{1}{x-1} \int_1^x \frac{e^{t^2}}{x+t}\, dt$$

Solution.

Well, that integral is a bit of a nightmare. Time for Calculus Thinking[1] to swing into action to start looking for a better way.

The signal really is in the way that improper integral is phrased. We sure would like to take the derivative of that thing, somehow, right? Well. Just to clarify, let's rewrite our given limit as:

$$L = \lim_{x\to 1} \frac{\int_1^x \frac{e^{t^2}}{x+t}\, dt}{x-1}$$

Since L is a limit, we can substitute 1 for x, giving us:

$$L = \lim_{x\to 1} \frac{\int_1^1 \frac{e^{t^2}}{1+t}\, dt}{1-1}$$

Focusing on the integral for a second, we see that the lower limit and upper limit are the same, which, by definition, means that the integral is zero, regardless of what the integrand is. The denominator is also clearly zero at $x = 1$, so we have a L'Hôpital's Rule situation! This grants us the ability take the derivative of that integral in t, taking it to an expression in x, via the Fundamental Theorem of Calculus. Sweet. If we take our fraction to be $\frac{f(x)}{g(x)}$, we can say:

$$f(x) = \int_1^x \frac{e^{t^2}}{x+t}\, dt \qquad \text{(numerator)}$$

$$f'(x) = \frac{e^{x^2}}{x+x} \qquad \text{(differentiate, apply the Fundamental Theorem of Calculus)}$$

[1]

$$f'(x) = \frac{e^{x^2}}{2x} \qquad \text{(simplify)}$$

$$g(x) = x - 1 \qquad \text{(denominator)}$$
$$g'(x) = 1 \qquad \text{(differentiate)}$$

Now, applying L'Hôpital's Rule, our limit becomes:

$$L = \lim_{x \to 1} \frac{\left(\frac{e^{x^2}}{2x}\right)}{1}$$

$$= \lim_{x \to 1} \frac{e^{x^2}}{2x}$$

$$= \frac{1}{2} \lim_{x \to 1} \frac{e^{x^2}}{x} \qquad \text{(factor out constant)}$$

$$= \frac{1}{2} \cdot \frac{e^{(1^2)}}{1} \qquad \text{(substitute)}$$

$$= \frac{1}{2} \cdot e \qquad \text{(simplify)}$$

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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Thank you so much.