A really interesting Calculus word problem: When should we start an aggressive vaccination program?

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Viruses and vaccinations are on everyone's minds these days, and calculus can help with some very specific problems. How important is the Fundamental Theorem of Calculus? How important is knowing how to use trigonometric identities?[1] This article will answer those questions and more.

Question.

Suppose the Health Department has been monitoring flu outbreaks continuously over the last 100 years and has found that the number of infections, $I(t)$, can be modeled by the function:

$$I(t) = cos\left(\frac{\pi}{6}t\right) + cos\left(\frac{\pi}{120}t\right) + 2$$

where t is in months and I is measured in $100,000$s of individuals. The Health Department wants to begin 5 years of intensive vaccination and quarantine at a time when the flu virus is at a 5-year minimum average. If we assume $t = 0$ is January 1, 2015, when should they begin the program?

Solution.

Reminder, the average value of a function, $f(t)$, over an interval $[a, b]$ is given by:

$$\frac{1}{b-a}\int_a^b f(t)\ dt \tag{1}$$

We are interested in the 5-year low / minimum average. So, we want to minimize:

$$\overline{I(t)} = \frac{1}{b-a}\int_a^b I(t)\ dt \tag{2}$$

over a 5-year interval. This means taking the derivative of something that looks like (2), and setting it equal to zero, basically, to find its critical points. It is a little more subtle than that, as we prepare to do the calculations, but this is the basis of the approach. A few interesting things will arise as we set things up and begin calculating. Get ready!

The first thing we have to keep in mind is that t is in months, and we are interested in a 5-year low. 5 years is $12 \cdot 5 = 60$ months. This will help us set the bounds for our integral, the a and b terms. We can see that the upper limit has to be whatever a is, plus 60, in order for us to get our target period's average function. So, our integral is really this - a function of a, the lower bound (the starting month):

$$\overline{I(a)} = \frac{1}{(a+60)-a}\int_a^{a+60} I(t)\ dt \tag{3}$$

[1] `https://en.wikipedia.org/wiki/List_of_trigonometric_identities` is the best resource for Trigonometric Identities, IMO.

$$\overline{I(a)} = \frac{1}{60}\int_a^{a+60} I(t)\ dt \tag{4}$$

Before we move on, let's take a closer look at (4), and see if we can't start to apply some Calculus Thinking[2]. We know that we need to find that starting value of a such that the average is at a 5-year low. We mentioned taking a derivative, above, and that is indeed the next step here. I want to do one more step to make things a little easier, and that is to multiply both sides by 60 before taking the derivative. I do this because all of the action is going to be on the right-hand side of the = sign, so I want to have that be as clear as possible. So:

$$\overline{I(a)} = \frac{1}{60}\int_a^{a+60} I(t)\ dt$$

$$60 \cdot \overline{I(a)} = \int_a^{a+60} I(t)\ dt \tag{5}$$

$$\frac{d}{da}\left[60 \cdot \overline{I(a)}\right] = \frac{d}{da}\left[\int_a^{a+60} I(t)\ dt\right] \tag{6}$$

Having a look at (6), it sure seems like we could use the Fundamental Theorem of Calculus to make our lives slightly easier. Remember that the Fundamental Theorem of Calculus allows us to take derivatives of integrals, and, since it establishes integration and differentiation as inverses, we can get away with not doing the integration at all. Recall this part of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_C^x f(t)\ dt = f(x) \tag{7}$$

for any constant, C.

The integral in (6) needs some massaging, however. And this is where we can do some cool trickery with the limits of integration. We need to have a constant - *any* constant, C, as our lower bound in order to use (7), above. Here is how we can do it. Rewrite the integral so that it consists of *two* integrals, like this, such that both have a constant as their lower bounds:

$$\frac{d}{da}\left[60 \cdot \overline{I(a)}\right] = \frac{d}{da}\left[\int_0^{a+60} I(t)\ dt - \int_0^{a} I(t)\ dt\right] \tag{8}$$

Take a moment to see that, using the properties of integrals, the difference of the two integrals in (8) is the same as the single integral in (6). *Now* we can get to differentiating:

$$60 \cdot \left[\overline{I(a)}\right]' = I(a+60) - I(a) \tag{9}$$

Cleaned right up, didn't it? Now we can solve, knowing that we are after the value of a that makes $\left[\overline{I(a)}\right]' = 0$ true:

$$\begin{aligned} 60 \cdot \left[\overline{I(a)}\right]' &= I(a+60) - I(a) \\ 60 \cdot 0 &= I(a+60) - I(a) \\ 0 &= I(a+60) - I(a) \end{aligned}$$

This shows that all we have to do is use our original function, $I(t)$ to find our critical points, a. Amazing! Let's go! **WARNING**: *Insane* trig identity gymnastics ahead!

[2] `https://mymathteacheristerrible.com/blog/2019/5/15/calculus-thinking-a-definition-i-use-over-and-over-with-my-students`

$$
\begin{aligned}
0 &= I(a+60) - I(a) \\
&= cos\left[\frac{\pi}{6}(a+60)\right] + cos\left[\frac{\pi}{120}(a+60)\right] + 2 - \left[cos\left(\frac{\pi}{6}a\right) + cos\left(\frac{\pi}{120}a\right) + 2\right] && \text{(substitute)} \\
&= cos\left[\frac{\pi}{6}(a+60)\right] + cos\left[\frac{\pi}{120}(a+60)\right] - \left[cos\left(\frac{\pi}{6}a\right) + cos\left(\frac{\pi}{120}a\right)\right] && \text{(2's cancel)} \\
&= cos\left[\frac{\pi}{6}(a+60)\right] + cos\left[\frac{\pi}{120}(a+60)\right] - cos\left(\frac{\pi}{6}a\right) - cos\left(\frac{\pi}{120}a\right) && \text{(distribute minus sign)} \\
&= cos\left[\frac{\pi}{6}a + \frac{\pi}{6}\cdot 60\right] + cos\left[\frac{\pi}{120}a + \frac{\pi}{120}\cdot 60\right] - cos\left(\frac{\pi}{6}a\right) - cos\left(\frac{\pi}{120}a\right) && \text{(expand angle expressions)} \\
&= cos\left[\frac{\pi}{6}a + 10\pi\right] + cos\left[\frac{\pi}{120}a + \frac{\pi}{2}\right] - cos\left(\frac{\pi}{6}a\right) - cos\left(\frac{\pi}{120}a\right) && \text{(simplify angle expressions)} \\
&= cos\left[\frac{\pi}{6}a\right] - sin\left[\frac{\pi}{120}a\right] - cos\left(\frac{\pi}{6}a\right) - cos\left(\frac{\pi}{120}a\right) && \text{(simplify w. trig identities)} \\
&= -sin\left(\frac{\pi}{120}a\right) - cos\left(\frac{\pi}{120}a\right) && \text{(simplify)} \\
0 &= sin\left(\frac{\pi}{120}a\right) + cos\left(\frac{\pi}{120}a\right) && \text{(divide by -1)} \\
0 &= sin^2\left(\frac{\pi}{120}a\right) + 2sin\left(\frac{\pi}{120}a\right)cos\left(\frac{\pi}{120}a\right) + cos^2\left(\frac{\pi}{120}a\right) && \text{(square both sides)} \\
&= 1 + 2sin\left(\frac{\pi}{120}a\right)cos\left(\frac{\pi}{120}a\right) && \text{(simplify, Pythagorean ident)} \\
&= 1 + sin2\left(\frac{\pi}{120}a\right) && \text{(simplify, double-angle ident)} \\
-1 &= sin2\left(\frac{\pi}{120}a\right) \\
\frac{3\pi}{2} + 2n\pi &= 2(\frac{\pi}{120}a) && \text{(inv. sine of both sides)} \\
\frac{3\pi}{4} + n\pi &= \frac{\pi}{120}a
\end{aligned}
$$

$90 + 120n = a$ (simplify)

Assuming the Health Department would want to get started as soon as possible, we can take n to be 0, leaving us with our answer of $a = 90$. That means, 90 days after $t = 0$ is when this program should start.

Like I said, some insane trig identity stuff in here, but it totally works. This goes to show just how important a all of that stuff is!

This is one of the best Calculus problems that I've worked on in a long time. I did verify the answer as being 90 days after $t = 0$.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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