Difficult Limits: Examples and Techniques

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1 Introduction

I began writing this handout with the intention of it being a quick and dirty approach to difficult limits problems in introductory Calculus. However, as I began to write about the various subtleties of the topic, I found the somewhat non-prescript character of the information here to need a bit more commentary than I originally intended. This handout is intended to slow down for a second, remind the reader that there are things they already know from 12-ish years of their mathematical career thus far, that it's still relevant, and consider what was in between the lines when your instructor covered this stuff in a brief hour or two. This entire handout is about the intuitive process of doing tricky algebra. Sometimes a change for new Calculus students, especially those fresh out of high school. Furthermore, **the concept of limits is the foundation upon which all of Calculus is built**! Why not explore it a little bit? Please trust that what I discuss here will be a useful 0^{th} or first step in the solution to many problems in your first 2 years of Calculus, and what makes this handout seem a little thick is the many lines of steps to solving example problems.

2 Basics & reminders

Let's begin with a cursory review of limits and limit rules. Generally, this is covered early in your Calculus textbook. Since this handout is more about difficult limit calculations, I'll just provide the end results of limit law proofs, so that we may use them to solve difficult limits problems straight away.

2.1 Basic limit rules

2.1.1 2-sided and 1-sided limits

There are 3 basic ways in which we consider limits:

1. limits as x approaches a from $x < a$
$$x \to a^-$$
2. limits as x approaches a from $x > a$
$$x \to a^+$$
3. limits as x approaches a
$$x \to a$$

Recall that

$$\lim_{x \to a} f(x) = L,$$

that is, L exists, iff (if, and only if)

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

2.1.2 Limit properties for arithmetic performed on functions: the foundations

There are a handful of basic ways to deal with limits of functions that are built out of arithmetic on other functions:

1. $$\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \tag{1}$$
Read as, "the limit of a sum is the sum of the limits,"
2. $$\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \tag{2}$$
Read as, "the limit of a product is the product of the limits,"
3. $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \tag{3}$$
Read as, "the limit of a quotient is the quotient of the limits," and
4. $$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}, \tag{4}$$
Read as, "the limit of the n^{th} root of a function is the n^{th} root of the limit."

For even values of n, please note that there is a domain restriction on the n^{th} root, which means there is ultimately a range restriction on $f(x)$ and a range restriction on $\lim_{x \to a} f(x)$.

2.1.3 Limits involving compositions of functions

It's a good idea to note here that there is a special property of limits that is very useful, since most real-world functions are not monomials related by basic arithmetic operations as above, but can be thought of as composite functions, that is, some $f(g(x))$ or something more complex.

Keep in mind that the reason we even bother to talk about combining functions in these ways at all in this context is that whenever the argument of a limit is anything other than a monomial (i.e., one term), the limit laws help us break up the argument into multiple limits that we can solve term-by-term, when needed. (I say, "when needed," because, sometimes, breaking things up makes the problem worse; this will be covered in this handout.) When we say $\lim_{x\to 5}[x+1]$ is 6, we are actually breaking up the argument using the sum rule, above. Specifically, $\lim_{x\to 5}[x+1] = \lim_{x\to 5}[x] + \lim_{x\to 5}[1] = 5+1 = 6$. Of course, the shorthand here is that we can use the original form, substitute $x = 5$ into the expression, then evaluate.

We will often creatively, using tricky algebra, break up the functions that compose the argument of a limit in order to "rephrase" the argument in ways that allow us to use algebra and the limit laws to calculate the limit. Many times in Calculus, there is a simpler problem or two "hiding out" in what's presented to us that allows us to turn a terrible-looking problem into tractable, easily-solvable subproblems based on first principles. I cannot stress this enough. I find that Calculus, in large part, is a practice of intuition and inference, more than it is a practice of actual calculations. Tricky limits problems is usually our first introduction to this kind of idea.

3 Difficult limits: Some issues

What happens when positing $\lim_{x\to a} f(x) = f(a)$ results in something undefined or difficult to deal with?

3.1 Limits at $\pm\infty$, substituting ∞ for x

For this topic, we must get very specific about $+\infty$ and $-\infty$ and the roles they play. $+\infty$ and $-\infty$ are not numbers. There are no calculations we can do on them. They are the upper and lower bounds of $\mathbb{R}$ (the set of all real numbers), placeholders to show that there is no smallest element of nor largest element of $\mathbb{R}$.

The roles of $+\infty$ and $-\infty$ can simply be recalled in the definition of any x being a real number, as for all $x \ \epsilon\ \mathbb{R}$, $x \ \epsilon\ (-\infty, +\infty)$. Please recall that when using interval notation, parentheses are used to show that the endpoints are not included in the interval. Using inequality notation, we can also say $-\infty\ <\ x\ <\ +\infty$, for all $x \ \epsilon\ \mathbb{R}$. If you find yourself saying, "I'll just substitute ∞ for x, ...," please take pause, for it's not an allowable operation.

Limits as $x \to \infty$ are, however, asking an important question that sometimes Calculus teachers fail to cover, and that is, "Does this function have an asymptote?" Asymptotes pop up everywhere, including rational functions, exponentials, inverse trig functions, etc. When you see a limit as $x \to \pm\infty$, ask yourself about asymptotes as you proceed.

3.2 Indeterminate forms, or, "L'ôpital's Rule is coming, but isn't here yet"

Just what are indeterminate forms? What do they look like? Here is a typical list:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty \pm \infty,\ 0 \times \infty,\ 0^0,\ 1^\infty,\ \infty^0$$

Please note that these forms are pretty specific. Here are some things that are *not* indeterminate forms:

1. Things like $1/0$ or any $n/0$ $(n \neq 0)$, $f(x)/0$, $(f(x) \neq 0)$, etc. are not indeterminate.
2. Limits that go to infinity. This still means that our function approaches infinity as $x \to a$ (i.e., grows without bound) and that's fine. This is the usefulness of the concept of infinity, above.

3. Limits that do not exist (DNE) are not indeterminate forms, either. Please refer to 1.1.1, above.

These are usually sources of confusion for new Calculus students. Please resist categorizing lots of things as indeterminate forms; I'd say they are kind of rare. More specifically, feeling like you've found an indeterminate form in your calculations should first prompt a question about your assumptions and calculations, like, "Did I do enough algebra here?" Additionally, you should always look at the graph of the function in question. If the graph appears to do something unexpected according to the statement of the problem, that's a strong hint that more algebra can be done to clarify just what is going on. Here is a simple example. At first blush, using straight substitution, $\lim_{x\to 0} \frac{x}{x}$ appears to produce 0/0. But does it, really? Of course not. Well, mostly not. Please recall some closely-related points about limits.

1. Implicit in all of our problem-solving approaches is **consideration of the function only on its domain**. This is one reason why learning how to specify domains for given functions is an important part of pre-Calculus courses.

2. A corollary here is that **we are only ever concerned with the behavior of a function super super super super super *close* to the value that x is approaching** (sometimes called a, δ-neighborhood) **in the limit we're evaluating, not *at* that value**. This mostly helps us with the, "Did I do enough algebra?" question, as we'll see.

3. Finally, **a limit is about where a function is *going*, not whether it ever *gets there***. Furthermore, the value that x approaches in a limit need not necessarily be in the domain of $f(x)$. In the case of, say, polynomials, $f(x)$ does both - it *goes* to a particular value, and is also able to be *evaluated* there, i.e., it *gets* there. Not every function behaves as straightforwardly as polynomials do.

In light of these points, let us reconsider $\lim_{x\to 0} \frac{x}{x}$. Recall from basic algebra and considerations involving rational functions, that a fraction is defined so long as the denominator is not zero. Furthermore, if we can guarantee that the denominator will never be zero, we can proceed with dividing the numerator by the denominator. Since we are not interested in the value of $\frac{x}{x}$ *at* zero, we can safely divide the x terms, leaving us with $\lim_{x\to 0} 1 = 1$. *Now* we can say we've done enough algebra. Please confirm with your graphing calculator or by hand that the graph of $y = x/x$ is indeed a horizontal line at $y = 1$ with a hole at $(0, 1)$.

3.3 Rational functions

I would venture to say that lots of the difficult limits problems we see in Calc I involve rational functions. There are simply too many things to mind, and numerators and denominators always seem to interact in unexpected ways. Throw a few trig functions in there, and things really begin to get weird sometimes. This is why a lot of PreCalc courses spend a lot of time on rational functions; they are notoriously tricky and sometimes things are hiding out in their definitions. With some algebra tricks, however, we can make lots of improvements and, step by step, deal with complicated-looking functions.

It is worth mentioning here that if we've worked with rationals before, we know a couple of things — limit-related things — about rational functions:

1. **Vertical asymptotes** are where the denominator of a rational function is zero. Specifically, $\lim_{x\to a} f(x)$, where a is a vertical asymptote is either $\pm\infty$ (function blows up in absolute value in the same direction on either side of the asymptote) or DNE (function blows up in different directions on either side of the asymptote). The trick of setting the denominator equal to zero and finding those values of x that make that true is still the technique here.

2. **Horizontal asymptotes** are simply $\lim_{x\to\pm\infty} f(x)$. Techniques like the leading coefficient test for rationals composed of polynomials, the conjugate trick, polynomial division, factoring and canceling, as well as rationalizing the denominator all still work here.

In the limits game, we must still take extra care with rationals.

4 Difficult limits: The techniques

4.1 Rephrasing, *un*-simplifying, or non-obvious tricks that really help

The main issue with this entire topic, and with a lot of Calculus (and possibly *Mathematics*, in general), is that there is not really a prescriptive technique for calculation. I find that new Calculus students often have an expectation that there are $1:1$ or $1:n$, for small n techniques for a particular pattern. This is unfortunate, and is likely a result of training, thus far, that has focused only on techniques, where the goal of solving problems on exams is focused on one or two techniques, for ease of understanding the problem, testing understanding, and, even more likely, easy grading. Calculus is a very different practice.

4.1.1 Double what you know! Read theorems and formulae forwards *and* backwards

I cannot stress this one enough. We are so used to seeing equations and theorems in only one direction. Consider a well-known expression:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b} \tag{5}$$

No doubt, you've used this many times, like:

$$\begin{aligned} \sqrt{8} &= \sqrt{2} \cdot \sqrt{4} \\ &= \sqrt{2} \cdot 2 \\ &= 2 \cdot \sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

What happens when we want to simplify something like, $2\sqrt{2} + \sqrt{24}$? Can we? Let's do it, using the product rule for radicals:

$$\begin{aligned} 2\sqrt{2} + \sqrt{24} &= 2\sqrt{2} + \sqrt{2}\sqrt{12} \\ &= \sqrt{2}(2 + \sqrt{12}) \\ &= \sqrt{2}(2 + \sqrt{4}\sqrt{3}) \\ &= \sqrt{2}(2 + 2\sqrt{3}) \\ &= 2\sqrt{2}(1 + \sqrt{3}) \end{aligned}$$

Not too bad. But if we read the product rule backwards, we can proceed a little quicker, or just differently, based on what we might see. Here, if we see 24 as $8 \cdot 3$, we might proceed:

$$\begin{aligned} 2\sqrt{2} + \sqrt{24} &= 2\sqrt{2} + \sqrt{8}\sqrt{3} && \text{(factor } \sqrt{24}\text{)} \\ &= (\sqrt{2})^2\sqrt{2} + \sqrt{8}\sqrt{3} && \text{(rewrite 2, cleverly)} \\ &= \sqrt{2}\sqrt{2}\sqrt{2} + \sqrt{8}\sqrt{3} && \text{(expand)} \\ &= \sqrt{8} + \sqrt{8}\sqrt{3} && \text{(use (11), backwards)} \\ &= \sqrt{8}(1 + \sqrt{3}) \\ &= 2\sqrt{2}(1 + \sqrt{3}) \end{aligned}$$

Of course, this problem is nearly nonsensical. But it does raise a simple point, and that is, algebra techniques are the core of dealing with tricky problems, and I believe this is one of the cores of doing calculations many times in Calculus. Another classic is, $sin^2x + cos^2x = 1$. Of course, whenever we see $sin^2x + cos^2x$ in an expression, we replace it with 1. But, you may remember from trig proofs, that sometimes, we replace 1 with $sin^2x + cos^2x$!

4.2 Rationals: factoring and cancelling

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

If we take this limit and substitute 2 for x, it would appear that we get an indeterminate form. This should prompt us to ask if we've done enough algebra. There is an additional signal, since this is a rational function with polynomials: see if we can factor the top and bottom, then see if anything cancels. Additionally, if possible, graph the function to see if it looks like we expect it to, if it's weird, or if it nudges us toward a technique we should use. Begin by factoring the top of the fraction. Of course, $x^2 - 4 = (x + 2)(x - 2)$, via the difference of squares pattern. Rewriting, we have:

$$\begin{aligned}
\lim_{x \to 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} && \text{(factored form)} \\
&= \lim_{x \to 2} (x + 2) && \text{(cancelled form)} \\
&= \lim_{x \to 2} x + \lim_{x \to 2} 2 && \text{(by (6))} \\
&= 2 + 2 \\
&= 4
\end{aligned}$$

Quite a bit different than 0/0, yes? Canceling and not worrying about holes, etc. is totally valid here, keeping in mind that calculating a limit is about where a function is going, rather than if it ever gets there.

4.3 Rationals: Asymptotes, limits at $\pm\infty$, and the keys to dealing with them

Earlier in this handout, I drew the parallel between limits at infinity and asymptotes. Here is the one limit that underlies the way we deal with most limits of rational functions, $x \to \infty$. Recall from algebra, that, for all $f(x) = \frac{1}{x^n}$, where n is a positive integer, $f(x)$ has an asymptote of 0. In terms of limits:

$$\lim_{x \to \infty} \frac{1}{x^n} = 0, \qquad n \in \mathbb{Z},\ n > 0 \tag{6}$$

This fact, along with the other techniques mentioned in this section, will allow us to handle tough-looking limits with ease, as we break things down into this form. In fact, when dealing with rationals, we will do all manner of algebraic manipulations in order to have components of our limits at infinity take on the form $1/x^n$. As a thought exercise, think about ways in which we might combine this idea with, say, (9), above.

4.4 Rationals: dealing with roots

Consider:

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x^4 + 1}}$$

This one is really interesting. The thing that makes it difficult to deal with is that we cannot do anything with the terms under the radical in the denominator. There simply is no operation we can do in its current state; a rephrase is needed.

We know this rule for working with fractions and radicals:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \tag{7}$$

In words, we can say, "the square root of a quotient is the quotient of the square roots." This says that we have two choices when dealing with that expression: we can divide a by b and take the square root, or we can "distribute" the radical to the numerator and denominator of the fraction, take the root of each, and *then* divide. Reading right to left,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \tag{8}$$

will help us if we are clever about rephrasing the numerator. We need a way to pull the entire fraction under the radical. If we think about how x^2 is the *result* of having taken a root, as in *this* classic *un*-simplifying,

$$\begin{aligned} x^2 &= \sqrt{(x^2)^2} \\ &= \sqrt{x^4} \end{aligned}$$

Then, we can rewrite and work:

$$\begin{aligned} \lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} &= \lim_{x\to\infty} \frac{\sqrt{x^4}}{\sqrt{x^4+1}} \\ &= \lim_{x\to\infty} \sqrt{\frac{x^4}{x^4+1}} && \text{by (8)} \\ &= \sqrt{\lim_{x\to\infty} \frac{x^4}{x^4+1}} && \text{by (4)} \\ &= \sqrt{1} && \text{(by the leading coefficient test)} \\ &= 1 \end{aligned}$$

4.5 Rationals: rational expressions mean, "divide," so let's get to it

Consider:

$$\lim_{x\to\infty} \frac{x^5+3x^4+1}{x^3-1}$$

This one is actually pretty straightforward. I recommend that if you can divide, you should at least *try* it. The least it will do is help visualize the graph of the function. One thing I feel like teachers don't emphasize enough is, for a rational expression $f(x)$ where $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, if we actually do the division, $f(x)$ is of the form $f(x) = A(x) + R(x)$, where $A(x)$ is the asymptote as $x \to \pm\infty$, and $R(x)$ is the remainder. $R(x)$ is a rational term whose denominator is $Q(x)$. I feel like, for many, prior to Calculus or more generalized math studies, asymptotes were positioned as *lines*. Lines are but one possibility.

After we do the division, our limit becomes:

$$\begin{aligned} \lim_{x\to\infty} \frac{x^5+3x^4+1}{x^3-1} &= \lim_{x\to\infty} [x^2+3x+\frac{x^2+3x+1}{x^3-1}] \\ &= \lim_{x\to\infty} [x^2+3x] + \lim_{x\to\infty} \frac{x^2+3x+1}{x^3-1} \\ &= \infty + 0 && \text{(by the leading coeff test)} \\ &= \infty \end{aligned}$$

4.6 Rationals: the conjugate trick

The essence of any use of the conjugate trick in algebra is the difference of squares pattern. Two conjugate binomials multiply to give the difference of their squares:

$$(a+b)(a-b) = a^2 - b^2 \tag{9}$$

Consider:

$$\lim_{x\to 1} \frac{2-\sqrt{3+x}}{x-1}$$

Straight substitution leads to 0/0, but did we do enough algebra? We really do not want the denominator, which evaluates to 0 with straight substitution, to sit there alone. We can multiply top and bottom of the fraction by the conjugate of $2-\sqrt{3+x}$, which is, $2+\sqrt{3+x}$:

$$\begin{aligned}
\lim_{x\to 1} \frac{2-\sqrt{3+x}}{x-1} &= \lim_{x\to 1} \frac{2-\sqrt{3+x}}{x-1} \cdot \frac{2+\sqrt{3+x}}{2+\sqrt{3+x}} \\
&= \lim_{x\to 1} \frac{4-(3+x)}{(x-1)(2+\sqrt{3+x})} \\
&= \lim_{x\to 1} \frac{-(x-1)}{(x-1)(2+\sqrt{3+x})} && \text{(expand and factor the numerator)} \\
&= \lim_{x\to 1} \frac{-1}{2+\sqrt{3+x}} && \text{(cancel terms)} \\
&= -\lim_{x\to 1} \frac{1}{2+\sqrt{3+x}} \\
&= -\frac{1}{4} && \text{(evaluate by direct substitution)}
\end{aligned}$$

Quite a bit different than 0/0, yes? Draw the graph of the original function to see that this limit is indeed correct.

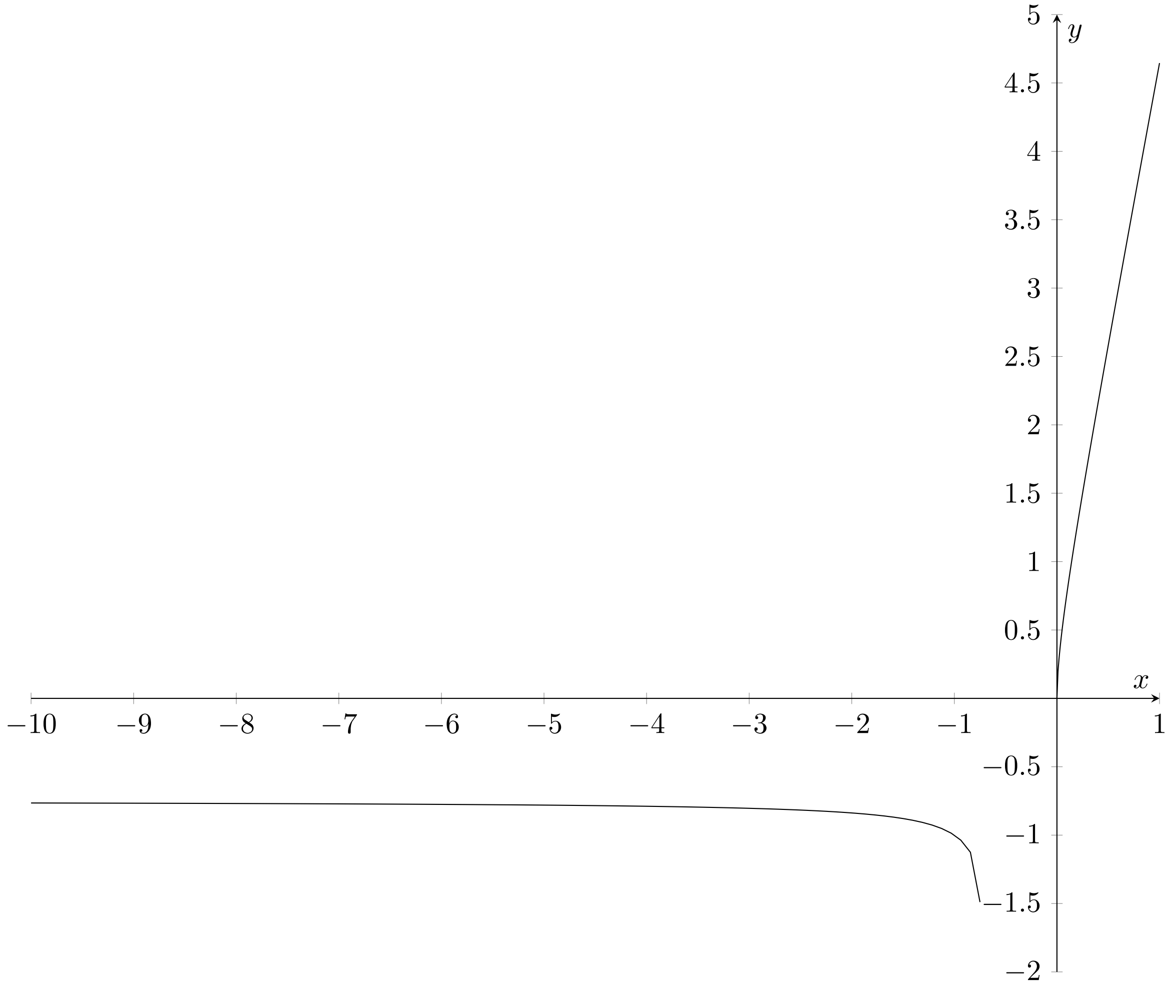
When dealing with expressions that have both polynomial terms and not, in particular, with radicals, some form of the conjugate trick might help, so definitely try it! It's hard to see these things hiding out in some expressions! I feel like I am surprised over and over.

4.7 Roots: *making* a rational expression(!)

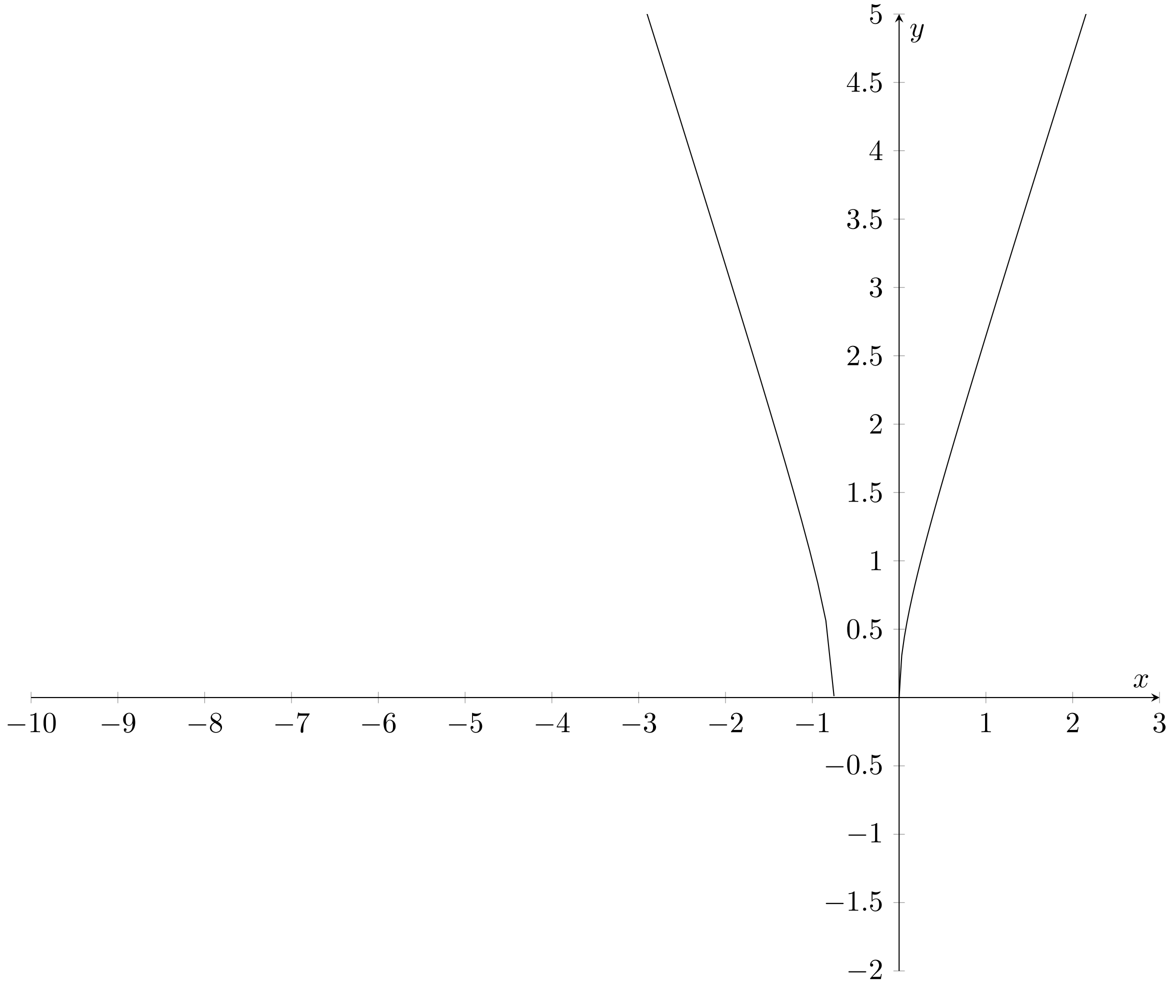
Consider:

$$\lim_{x\to -\infty} \sqrt{4x^2+3x} + 2x$$

This one is super tough. I recommend having a look at the graph, if you can. Here it is, for reference.



From the picture, we get the idea that perhaps there is an asymptote in the range $-1 < y < 0$, as $x \to -\infty$. Clear from the picture, but not very obvious from the expression $\sqrt{4x^2+3x}+2x$. Furthermore, a picture doesn't help much with algebra. Thinking algebraically, the square root presents a real problem here. And it's not obvious how much the $+2x$ term contributes to where the asymptote is. Here is the same graph, but without the $+2x$ term.



It's probably easier to reason about the shape of the graph for $x > 0$, but for $x \to -\infty$, it's tough. In this case, viewing our limit expression as having potential for the conjugate trick (4.6) helps here. In yet another *un*-simplifying move, we *create* a fraction where there was not one to begin with.

$$\begin{aligned}
\lim_{x\to-\infty} \sqrt{4x^2+3x}+2x &= \lim_{x\to-\infty} \frac{\sqrt{4x^2+3x}+2x}{1} \\
&= \lim_{x\to-\infty} \frac{\sqrt{4x^2+3x}+2x}{1} \cdot \frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x} \\
&= \lim_{x\to-\infty} \frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x}-2x} \\
&= 3 \cdot \lim_{x\to-\infty} \frac{x}{\sqrt{4x^2+3x}-2x}
\end{aligned}$$

Probably doesn't look much better, I agree, but we could calculate this limit at $-\infty$:

$$3 \cdot \lim_{x \to -\infty} \frac{x}{\sqrt{4x^2+3x} - 2x} = 3 \cdot \lim_{x \to -\infty} \frac{1}{\sqrt{\frac{4x^2+3x}{x^2}} - 2}$$

$$= 3 \cdot \lim_{x \to -\infty} \frac{1}{\sqrt{4 + \frac{3}{x}} - 2}$$

$$= 3 \cdot \frac{1}{0}$$

Still not good, but there is yet another trick.

Since our function $\frac{x}{\sqrt{4x^2+3x}-2x}$ is defined for all x as we head to $-\infty$, we really don't have to concern ourselves with domain exceptions. Specifically, our function gives a zero denominator at $x = 0$, but $-\infty$ is so far away from zero, that it does not matter; what we seek is an asymptote as $x \to -\infty$. So, we can continue on with some of the techniques we've already explored, plus an interesting factoring trick. Since the numerator and denominator of our function both contain x, we can divide top and bottom by x, leaving us with:

$$3 \cdot \lim_{x \to -\infty} \frac{x}{\sqrt{4x^2+3x} - 2x} = 3 \cdot \lim_{x \to -\infty} \frac{1}{\frac{1}{x}\sqrt{4x^2+3x} - 2}$$

Now, for a trick with the radical. We can factor underneath the radical, giving us:

$$\sqrt{4x^2+3x} = \sqrt{x^2\left(4 + \frac{3}{x}\right)}$$

$$= \sqrt{x^2} \cdot \sqrt{4 + \frac{3}{x}}$$

The $\sqrt{x^2}$ term is curious, in this context. Squaring something always results in a positive number, so it sort of obscures what our intial value of x was. The radical without a sign also obscures what's happening by only giving the positive root. Since we know that the only domain exception for our function is $x = 0$, we have to somehow *preserve* the sign of x in our calculations, such that it allows us to say whether the number is positive or negative, depending on where we are in the function's domain. A great tool for this is *absolute value*. So our expression for the factored radical in the denominator will become:

$$\sqrt{x^2} \cdot \sqrt{4 + \frac{3}{x}} = |x| \cdot \sqrt{4 + \frac{3}{x}}$$

The use of absolute value here allows us to preserve the sign of x. Super cool. Now we can continue and show how this trick affects evaluating the limit properly. We have:

$$3 \cdot \lim_{x \to -\infty} \frac{1}{\frac{1}{x}\sqrt{4x^2+3x} - 2} = 3 \cdot \lim_{x \to -\infty} \frac{1}{\frac{1}{x}\sqrt{x^2} \cdot \sqrt{4 + \frac{3}{x}} - 2}$$

$$= 3 \cdot \lim_{x \to -\infty} \frac{1}{\frac{|x|}{x} \cdot \sqrt{4 + \frac{3}{x}} - 2} \qquad \text{(take the square root, use absolute value)}$$

Almost there. Now, let's have a look at $\frac{|x|}{x}$. This is a piecewise step function:

$$\frac{|x|}{x} = \begin{cases} 1, & x < 0 \\ -1, & x > 0 \end{cases}$$

And, since we are headed to $-\infty$, we can simply replace the $\frac{|x|}{x}$ term with -1, giving us:

$$\begin{aligned} 3 \cdot \lim_{x \to -\infty} \frac{1}{\frac{|x|}{x} \cdot \sqrt{4 + \frac{3}{x}} - 2} &= 3 \cdot \lim_{x \to -\infty} \frac{1}{-\sqrt{4 + \frac{3}{x}} - 2} \\ &= 3 \cdot \frac{1}{-\sqrt{4} - 2} \\ &= -\frac{3}{4} \end{aligned}$$

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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