An interesting conic sections (parabola) problem

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Conic sections problems can be tricky for many reasons. In this problem, we are only given some points and the parabola's orientation, and we are asked to find the equation of the parabola in vertex form. This article demonstrates two approaches to solving the problem.

Question.

Given 3 points, $(-3, 3)$, $(-4, 3)$, and $(-9, 27)$, find the vertex form of a parabola, if the parabola opens straight up or straight down.

Solution.

At first glance, there does not appear to be a lot of information here, yet, the information given is really the minimum amount of detail we would need to solve this problem. On your own, convince yourself that a unique parabola can be drawn through 3 given points. (Suggestion: try drawing parabolas through two points, and then through only one point, and see if the parabolas drawn are indeed *unique* - this is the key bit.)

Once we consider the basic idea here, we can use the "... if the parabola opens straight up or down" detail. As a reminder, a parabola that opens straight up or straight down, in vertex form, is:

$$(x - h)^2 = 4p(y - k)$$

First thing I tried was to substitute all the points into this form and see where that left me. It left me with too many variables to solve for, using any *simple* means. I'd have a quadratic in h, and a linear term involving p and k. Do-able maybe with Desmos[1] , but not ideal.

But! Just because we are *asked* for this form in our *final answer*, it does not mean we have to *start* there. Recall the *standard form* of a parabola:

$$y = Ax^2 + Bx + C$$

This is a *way* better place to start, because, when we substitute the x and y values for all three of the given points, all the troublesome terms (the square terms, most importantly), go away, as in:

$$3 = A(-3)^2 + B(-3) + C \qquad \text{[for point (-3,3)]}$$
$$3 = A(-4)^2 + B(-4) + C \qquad \text{[for point (-4,3)]}$$
$$27 = A(-9)^2 + B(-9) + C \qquad \text{[for point (-9,27)]}$$

[1] `https://desmos.com/calculator`

Reorganizing a bit, we get a system of 3 equations in 3 variables:

$$9A - 3B + C = 3 \tag{1}$$
$$16A - 4B + C = 3 \tag{2}$$
$$81A - 9B + C = 27 \tag{3}$$

If you are taking, say, a high school math class that doesn't cover how to handle this situation, my general comment is that we can handle this using elimination, just as we do for a system of 2 equations in two variables, with which you should be familiar. The signal here is in (1) and (2): both have 3's on the right-hand side of the equals sign, and, more importantly, those C's on the left. We can start with those two equations:

$$16A - 4B + C = 3$$
$$9A - 3B + C = 3$$

Subtracting the bottom equation from the top equation leaves us:

$$7A - B = 0$$
$$B = 7A \qquad \text{(solving for } B\text{)}$$

We did elimination to get B, but now, we can use *substitution* to simplify our system. When we do that, the resulting system becomes:

$$9A - 3(7A) + C = 3$$
$$16A - 4(7A) + C = 3$$
$$81A - 9(7A) + C = 27$$

Simplifying:

$$9A - 21A + C = 3$$
$$16A - 28A + C = 3$$
$$81A - 63A + C = 27$$

Then:

$$-12A + C = 3 \tag{4}$$
$$-12A + C = 3 \tag{5}$$
$$18A + C = 27 \tag{6}$$

This leaves us two equations that are redundant, so we can take one of those and the third equation and use elimination *again*:

$$-12A + C = 3$$
$$18A + C = 27$$

Subtracting the top equation from the bottom equation gives us:

$$30A = 24$$

$$A = \frac{24}{30} = \frac{4}{5}$$

Since we now know $A = \frac{4}{5}$, we can also use $B = 7A$ to say that $B = \frac{28}{5}$. We can use these values in any of our 3 equations to arrive at $C = \frac{63}{5}$. This leaves us with

$$y = \frac{4}{5}x^2 + \frac{28}{5}x + \frac{63}{5} \tag{7}$$

Now, we have to convert this standard-form expression for this parabola to vertex form:

$$(x - h)^2 = 4p(y - k)$$

In general, the technique we use in order to do this is *completing the square.* In preparation for that, we can rearrange (7), such that the x terms are on one side of the equation:

$$y - \frac{63}{5} = \frac{4}{5}x^2 + \frac{28}{5}x$$

Multiplying both sides by 5 gets us a slightly neater-looking equation:

$$5\left(y - \frac{63}{5}\right) = 4x^2 + 28x$$

We need the coefficient of the x^2 term to be 1, so we should divide both sides by 4:

$$\frac{5}{4}\left(y - \frac{63}{5}\right) = x^2 + 7x \tag{8}$$

Next, the technique tells us to take the b term, 7, divide it by 2, square it, and add to both sides.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{7}{2}\right)^2$$

$$= \frac{49}{4}$$

Adding this value to both sides of (8), we get:

$$\frac{5}{4}\left(y - \frac{63}{5}\right) + \frac{49}{4} = x^2 + 7x + \frac{49}{4}$$

Factoring the right-hand side to yield our perfect square, we get:

$$\frac{5}{4}\left(y - \frac{63}{5}\right) + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$$

The left-hand side is a bit messy, yet, so we have to distribute the $\frac{5}{4}$ and combine like terms:

$$\frac{5}{4}y - \frac{315}{20} + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2 \qquad \text{(distribute)}$$

$$\frac{5}{4}y - \frac{63}{4} + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2 \qquad \text{(reduce middle fraction)}$$

$$\frac{5}{4}y - \frac{14}{4} = \left(x + \frac{7}{2}\right)^2 \qquad \text{(combine like terms)}$$

$$\frac{5}{4}\left(y-\frac{14}{5}\right)=\left(x+\frac{7}{2}\right)^2 \qquad \text{(factor to isolate the } y \text{ and the } 4p \text{ term)}$$

Done!

Now. There is a super interesting shortcut here, if we are able to see how the given information can affect working directly with the vertex form of this parabola. The question states that the parabola is oriented either up or down, that is, *the axis of symmetry is parallel to the y-axis.* The additional hint here is that we have two points with the same y-coordinate and two different x-coordinates. Remind yourself that, for any parabola, as defined in this question, two points with the same y-coordinate lie on a line segment whose perpendicular bisector is the axis of symmetry. Furthermore, the equation for the axis of symmetry of this parabola is $x = h$. Recalling from Geometry that we can take the average of the x-coordinates to calculate the midpoint of the segment $(-4, 3)$ to $(-3, 3)$ (draw yourself a picture!), we can say $h = \frac{(-3+(-4))}{2} = -\frac{7}{2}$, making what we know so far about the vertex form of our parabola:

$$\left(x+\frac{7}{2}\right)^2 = 4p(y-k)$$

Doesn't look *too* much better, but! It's *better enough.* Let's have a closer look. The expression above has 4 variables. But, we have (x, y) pairs in the question, so we can substitute those in, leaving us with only p and k as unknowns. All we have to do is take two of the given points and make a system of equations in two variables to solve for p and k, and we'll be done. If we use $(-3, 3)$ and $(-9, 27)$:

$$\left(-3+\frac{7}{2}\right)^2 = 4p(3-k) \qquad [\text{for point } (-3,3)] \tag{9}$$

$$\left(-9+\frac{7}{2}\right)^2 = 4p(27-k) \qquad [\text{for point } (-9,27)] \tag{10}$$

Dealing with the numbers we can calculate on the left-hand sides of both equations:

$$\frac{1}{4} = 4p(3-k)$$

$$\frac{121}{4} = 4p(27-k)$$

We can divide each equation by 4, leaving us with:

$$\frac{1}{16} = p(3-k)$$

$$\frac{121}{16} = p(27-k)$$

And now what we can do with these two equations is, *divide* them. This is how we perform elimination of p in both equations, leaving us with a proportion:

$$\frac{\frac{1}{16}}{\frac{121}{16}} = \frac{3-k}{27-k}$$

$$\frac{1}{121} = \frac{3-k}{27-k} \qquad \text{(simplify compound fraction on the left-hand side)}$$

We have to add the restriction, $k \neq 27$, to solve this porportion. We can then solve for k by cross multiplication:

$$\begin{aligned} 1 \cdot (27-k) &= 121(3-k) \\ 27-k &= 363-121k \\ 120k &= 336 \\ k &= \frac{336}{120} = \frac{14}{5} \end{aligned}$$

Leaving us with

$$\left(x+\frac{7}{2}\right)^2 = 4p\left(y-\frac{14}{5}\right)$$

as our vertex form, so far. One more thing to do: solve for p. This is a simple matter, if we use any one of the given points. Using point $(-4,3)$:

$$\left(x+\frac{7}{2}\right)^2 = 4p\left(y-\frac{14}{5}\right)$$

$$\left(-4+\frac{7}{2}\right)^2 = 4p\left(3-\frac{14}{5}\right)$$

$$\left(-\frac{1}{2}\right)^2 = 4p\left(\frac{1}{5}\right)$$

$$\frac{1}{4} = \frac{4}{5}p$$

$$5 \cdot \frac{1}{4} = 4p$$

$$\frac{5}{4} = 4p \qquad \text{(we really are interested in } 4p\text{! Stop.)}$$

Now we have all of the required terms to formulate the vertex form of our parabola!

$$\left(x+\frac{7}{2}\right)^2 = \frac{5}{4}\left(y-\frac{14}{5}\right)$$

Done, having used another method.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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