A cool second-order differential equation problem

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Having a look at the machinations of how solving problems works sometimes results in a deeper understanding of what solving certain types of problems is all about. This differential equation problem is interesting, in that regard.

Question.

The function $y(t) = e^{-2t} cos\, t - e^{-2t} sin\, t$, $t_0 = 0$ is a solution of the initial value problem $y'' + ay' + by = 0$, where $y(t_0) = y_0$ and $y'(t_0) = y'_0$. Determine the constants a, b, y_0, and y'_0.

Solution.

As I mention in the abstract, this problem is really interesting. We could, of course, go *right* at it and begin differentiating $y(t)$ straight away and see what happens, or(!), better yet, we can take a step back, engage in some *Calculus Thinking*,[1] and see how we might subvert a brute force approach...

So, what do we know? This appears to be a homogeneous, 2nd order differential equation. At the core of solving these kinds of problems is some sort of characteristic equation, as well as how that characteristic equation impacts the format of the solution to the initial value problem. Easy enough. What this problem is really asking us to do is view solving it from a perspective where we're sort of looking backwards from the provided $y(t)$. That being said, from the problem statement, we know our characteristic equation is:

$$r^2 + ar + b = 0$$

This is a fairly simply-stated quadratic.

And there's more. Note that the problem statement actually gives us a full $y(t)$ solution. Please also recall that the *format* of the solution tells us a lot about the roots of the characteristic equation. Three choices, but we need to consider them somewhat backwards from the way we usually do. For any characteristic equation, $r^2 + br + c = 0$ (for constants b and c), the roots/solutions are:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad \text{(2 real roots: } r_1,\ r_2) \tag{1}$$
$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t} \qquad \text{(1 repeated root: } r_1) \tag{2}$$
$$y(t) = c_1 e^{\alpha t} sin(\beta t) + c_2 e^{\alpha t} cos(\beta t) \qquad \text{(complex roots: } r_{1,2} = \alpha \pm \beta i) \tag{3}$$

Looking at the choices, we clearly have a given solution that looks like (3). Continuing to fill in our picture using the given information, we know:

$$c_1 = 1$$
$$c_2 = -1$$

[1] https://mymathteacheristerrible.com/blog/2019/5/15/calculus-thinking-a-definition-i-use-over-and-over-with-my-students

$$
\begin{aligned}
\alpha &= -2 \\
\beta &= 1
\end{aligned}
$$

Most importantly, with this info, we can construct what our complex roots are, using α and β:

$$
\begin{aligned}
r_{1,2} &= \alpha \pm \beta i && (4) \\
&= -2 \pm i && (5)
\end{aligned}
$$

Back to Algebra I or II for a moment. If we have complex roots, we should be able to calculate the missing constants in the characteristic equation for our given differential equation. The best approach is probably to go back to the quadratic formula. Note that there is a potential confusion tactic in the way the differential equation is stated; potential, because we're not going to fall for it here! Recall the quadratic formula for a standard-form quadratic equation $ar^2 + br + c = 0$:

$$
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$

For our characteristic equation, $r^2 + ar + b = 0$, we have to be careful with the coefficients as they are labeled. In our equation, we know that the coefficient of the squared term is 1, making our roots:

$$
r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4(1)b}}{2(1)}
$$

$$
r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}
$$

From (3), we have:

$$
\begin{aligned}
-2 \pm i &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} && \text{(substitute, using (5))} \\
-2 \pm i &= \frac{-a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2} && \text{(rearrange to make next steps easier)}
\end{aligned}
$$

We can now separate the real and imaginary parts, giving us:

$$
\begin{aligned}
-2 &= -\frac{a}{2} \\
a &= 4 && (6)
\end{aligned}
$$

and:

$$
i = \frac{\sqrt{a^2 - 4b}}{2}
$$

$$
\begin{aligned}
2i &= \sqrt{a^2 - 4b} \\
-4 &= a^2 - 4b \\
-4 &= 4^2 - 4b && \text{(substitute, using (6))} \\
-1 &= 4 - b && \text{(divide by 4)} \\
b &= 5
\end{aligned}
$$

making our characteristic equation:

$$
r^2 + 4r + 5 = 0
$$

and, therefore, our differential equation:

$$y'' + 4y' + 5y = 0 \tag{7}$$

Not bad! But, we're not done. This is (also) an initial value problem, so, we have to deal with said initial values. We're given $t_0 = 0$, $y(t_0) = y_0$, and $y'(t_0) = y'_0$. It's a bit of a weirdly-stated set of conditions, but if we do some substituting, we can come up with initial conditions involving y and y':

$$\begin{aligned} y(t_0) &= y_0 \\ y(0) &= y_0 \end{aligned} \tag{8}$$

$$\begin{aligned} y'(t_0) &= y'_0 \\ y'(0) &= y'_0 \end{aligned} \tag{9}$$

Note that all of the values on the left-hand side of the above are all *constants*. I mention this because something like y'_0 is a bit nonstandard, as there really is no *derivative* there, rather, it is the *value* of the derivative of $y(t)$ at $t = 0$.

Now that we have two expressions for the initial conditions, we need to take a derivative:

$$\begin{aligned} y(t) &= e^{-2t} cos\ t - e^{-2t} sin\ t \\ y'(t) &= -2e^{-2t} cos\ t - e^{-2t} sin\ t + 2e^{-2t} sin\ t - e^{-2t} cos\ t \\ &= -3e^{-2t} cos\ t + e^{-2t} sin\ t \end{aligned}$$

Now, we can deal with our initial conditions, using (8) and (9):

$$\begin{aligned} y(t) &= e^{-2t} cos\ t - e^{-2t} sin\ t \\ y_0 &= e^{-2(0)} cos(0) - e^{-2(0)} sin(0) \\ y_0 &= 1 \end{aligned}$$

$$\begin{aligned} y'(t) &= -3e^{-2t} cos\ t + e^{-2t} sin\ t \\ y'_0 &= -3e^{-2(0)} cos(0) + e^{-2(0)} sin(0) \\ y'_0 &= -3 \end{aligned}$$

And, with that, we have solved our differential equation problem!

To recap: $a = 4$, $b = 5$, $y_0 = 1$, and $y'_0 = -3$.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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