Solving a 2nd order ordinary differential equation using Variation of Parameters

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Here is a problem from a first-year DiffEq class with a straightforward solution.

Question.

Given $y''+4ty'+(2+4t^2)y = t^2e^{-t^2}$, with $y_1(t) = e^{-t^2}$ and $y_2(t) = te^{-t^2}$ as solutions to the homogeneous equation, determine: The complementary solution, $y_c(t)$, the particular solution, $y_p(t)$, using variation of parameters, and form the general solution, $y(t)$.

Solution.

For simplicity, let's break the solution into sections and build our solution step-by-step. We have a second-order ordinary differential equation of the form: $y'' + p(t)y' + q(t)y = g(t)$, where we are going to use variation of parameters.

1 The complementary solution, $y_c(t)$

This is the easiest part. To form the complementary solution, $y_c(t)$, all we need to do is take the given $y_1(t)$ and $y_2(t)$ and form a linear combination, using the superposition principle, so:

$$\begin{aligned} y_c(t) &= c_1y_1(t) + c_2y_2(t) \\ &= c_1e^{-t^2} + c_2te^{-t^2} && \text{(substitute)} \end{aligned} \tag{1}$$

2 The particular solution, $y_p(t)$, using variation of parameters

Here, we start by creating a linear combination of two functions u_1 and u_2, each multiplied by their corresponding $y_n(t)$, as

$$y_p(t) = u_1y_1 + u_2y_2$$

where u_n and y_n are functions of t. u_1 and u_2 are how we are varying y_1 and y_2, respectively.

We will also need the Wronskian, $W(y_1, y_2) = y_1y_2' - y_2y_1'$, and as we go through the process, we will wind up with two key things to help us find $u_1(t)$ and $u_2(t)$:

$$u_1' = \frac{-y_2 \cdot g(t)}{W(y_1, y_2)} \tag{2}$$

$$u_2' = \frac{y_1 \cdot g(t)}{W(y_1, y_2)} \tag{3}$$

Noting that our $g(t)$ here is $g(t) = t^2 e^{-t^2}$, we have:

$$u_1' = \frac{-y_2 \cdot g(t)}{W(y_1, y_2)}$$

$$u_1' = \frac{-te^{-t^2} t^2 e^{-t^2}}{W(y_1, y_2)} \qquad \text{(substitute)}$$

$$u_1' = \frac{-t^3 e^{-2t^2}}{W(y_1, y_2)} \qquad \text{(simplify)} \tag{4}$$

and:

$$u_2' = \frac{y_1 \cdot g(t)}{W(y_1, y_2)}$$

$$u_2' = \frac{e^{-t^2} t^2 e^{-t^2}}{W(y_1, y_2)} \qquad \text{(substitute)}$$

$$u_2' = \frac{t^2 e^{-2t^2}}{W(y_1, y_2)} \qquad \text{(simplify)} \tag{5}$$

Calculating the Wronskian, $W(y_1, y_2)$, we have:

$$\begin{aligned}
W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\
&= e^{-t^2} \cdot y_2' - e^{-t^2} \cdot y_1' \\
&= e^{-t^2}(e^{-t^2} - 2t^2 e^{-t^2}) - te^{-t^2}(-2te^{-t^2}) \\
&= e^{-2t^2} - 2t^2 e^{-2t^2} + 2t^2 e^{-2t^2} \\
W(y_1, y_2) &= e^{-2t^2}
\end{aligned} \tag{6}$$

Now, we have to integrate (4) and (5) to resolve what $u_1(t)$ and $u_2(t)$ are, making use of (6):

$$\begin{aligned}
u_1(t) &= \int u_1'(t)\ dt \\
&= \int \frac{-t^3 e^{-2t^2}}{W(y_1, y_2)}\ dt \\
&= \int \frac{-t^3 e^{-2t^2}}{e^{-2t^2}}\ dt && \text{(substitute)} \\
&= -\int t^3\ dt && \text{(simplify)} \\
u_1(t) &= -\frac{1}{4} t^4 && \text{(integrate)}
\end{aligned} \tag{7}$$

and:

$$
\begin{aligned}
u_2(t) &= \int u_2'(t)\ dt \\
&= \int \frac{t^2 e^{-2t^2}}{W(y_1, y_2)}\ dt \\
&= \int \frac{t^2 e^{-2t^2}}{e^{-2t^2}}\ dt && \text{(substitute)} \\
&= \int t^2\ dt && \text{(simplify)} \\
u_2(t) &= \frac{1}{3}t^3 && \text{(integrate)} \qquad (8)
\end{aligned}
$$

This makes our particular solution, $y_p(t)$:

$$
\begin{aligned}
y_p(t) &= u_1 y_1 + u_2 y_2 \\
&= -\frac{1}{4}t^4 e^{-t^2} + \frac{1}{3}t^3 \cdot te^{-t^2} && \text{(substitute)} \\
&= -\frac{1}{4}t^4 e^{-t^2} + \frac{1}{3}t^4 e^{-t^2} && \text{(simplify)} \\
y_p(t) &= \frac{1}{12}t^4 e^{-t^2} && \text{(combine like terms)} \qquad (9)
\end{aligned}
$$

3 The general solution, $y(t)$

The general solution to our differential equation, $y(t)$, is simply a linear combination of $y_1(t)$, $y_2(t)$, and $y_p(t)$:

$$
\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= c_1 y_1(t) + c_2 y_2(t) + y_p(t) \\
&= c_1 e^{-t^2} + c_2 te^{-t^2} + \frac{1}{12}t^4 e^{-t^2} \qquad (10)
\end{aligned}
$$

And, with (10), we are done!

Done.

Reporting errors and giving feedback

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